SLOW MAPPINGS OF FINITE DISTORTION

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Abstract. We examine mappings of finite distortion from Euclidean spaces into Riemannian manifolds. We use integral type isoperimetric inequalities to obtain Liouville type growth results under mild assumptions on the distortion of the mappings and the geometry of the manifolds.

1. Introduction

According to the classical Euclidean theory of quasiregular mappings *bounded entire quasiregular mappings are constant.* A mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$ in the Sobolev space $W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ is *K-quasiregular* if

$$\|Df\|^n \leqslant K J_f \text{ a.e.},$$

where $\|Df\|$ is the operator norm of the tangent map of f and J_f is the Jacobian determinant.

This version of the classical Liouville's theorem can be derived from the following qualitative lower growth bound estimate: *Given $n \geqslant 2$ and $K \geqslant 1$ there exists a constant $\alpha > 0$ depending only on n and K so that every K-quasiregular mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfying*

$$\lim_{|x|\to\infty} |x|^{-\alpha}|f(x)| = 0$$

is constant [18, III.1.13].

This growth result of Liouville type admit far reaching generalizations for quasiregular mappings into closed manifolds. Let $\mathbb{Y}$ be a closed, connected, and oriented Riemannian n-manifold receiving a non-constant K-quasiregular mapping f from $\mathbb{R}^n$. On the one hand, a theorem of Varopoulos [19, pp. 146-147] states that *the fundamental group of $\mathbb{Y}$ has the growth of order at most n.* On the other hand, a theorem of Bonk and Heinonen [1, Theorem 1.11] yields a lower growth bound: *if $\mathbb{Y}$ is not a rational homology sphere then*

$$\liminf_{r\to\infty} \frac{1}{r^\alpha} \int_{B^n(r)} J_f > 0, \tag{1.1}$$

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where $\alpha > 0$ depends only on n and K. Here and in what follows $B^n(r)$ stands for the open Euclidean n-ball centered at the origin of radius r. We also say that a smooth n-manifold $\mathbb{Y}$ is a *rational homology sphere* if it has the same *de Rham cohomology ring* $H^*(\mathbb{Y})$ as the standard n-sphere $\mathbb{S}^n$; in what follows, we denote the ℓth de Rham cohomology group of $\mathbb{Y}$ by $H^\ell(\mathbb{Y})$.

In this paper we show that both of these theorems hold for a wider class of mappings termed *mappings of finite distortion* and we give an interpretation of the theorems through isoperimetric inequalities for mappings. To be more precise, we consider continuous Sobolev mappings $f\colon \mathbb{R}^n \to \mathbb{Y}$ in $W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{Y})$, where $\mathbb{Y}$ is a connected and oriented Riemannian n-manifold. A mapping $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{Y})$ has *finite distortion* provided there exists a measurable function $K\colon \mathbb{R}^n \to [1,\infty)$ such that

$$\|Df(x)\|^n \leqslant K(x) J_f(x) \text{ for a.e. } x \in \mathbb{R}^n. \tag{1.2}$$

This class of mappings is intriguing. Such mappings may be seen as natural generalizations of quasiregular mappings. Indeed, under a certain integrability condition on the distortion function, mappings of finite distortion have geometric and topological behavior similar to quasiregular mappings; see the book by Iwaniec and Martin [11], and the references there.

According to our first theorem here, *mappings of p-mean distortion, $p > n-1$,* exhibit a similar growth rate as quasiregular mappings. This result generalizes the theorem of Bonk and Heinonen. We say that f has *mean distortion in L^p*, $1 \leqslant p < \infty$, if

$$\mathcal{K}_p = \mathcal{K}_{f,p} = \sup_{r \geqslant 1} \left(\fint_{B^n(r)} K_f(x)^p \,\mathrm{d}x \right)^{1/p} < \infty. \tag{1.3}$$

Theorem 1. *Let $\mathbb{Y}$ be a closed, connected, and oriented Riemannian n-manifold, $n \geqslant 2$, and $f\colon \mathbb{R}^n \to \mathbb{Y}$ a non-constant slow mapping of finite distortion with mean distortion in L^{n-1}. Then $\dim H^\ell(\mathbb{Y}) = 0$ for $1 < \ell < n-1$. If f has mean distortion in L^p for some $p > n-1$, then $\mathbb{Y}$ is a rational homology sphere, i.e., $\dim H^\ell(\mathbb{Y}) = 0$ for $1 \leqslant \ell \leqslant n-1$ and $\dim H^\ell(\mathbb{Y}) = 1$ for $\ell = 0, n$.*

We say that a mapping of finite distortion $f\colon \mathbb{R}^n \to \mathbb{Y}$ is *slow* if

$$\lim_{r\to\infty} \frac{1}{r^\alpha} \int_{B^n(r)} J_f = 0 \tag{1.4}$$

for every $\alpha > 0$.

Our second theorem gives an interpretation of the aforementioned Varopoulos's theorem as an end point in a spectrum of growth results. We say that a mapping of finite distortion $f\colon \mathbb{R}^n \to \mathbb{Y}$ has *at least logarithmic growth* if

$$\liminf_{r\to\infty} \frac{1}{(\log r)^\alpha} \int_{B^n(r)} J_f > 0$$

for some $\alpha > 0$.

Theorem 2. *Let $\mathbb{Y}$ be an open and oriented Riemannian n-manifold, $n \geqslant 2$, supporting a d-dimensional isoperimetric inequality, $d \geqslant 2$, and $f\colon \mathbb{R}^n \to \mathbb{Y}$ a mapping of finite distortion with mean distortion in L^{n-1}. Then f is*

(a) *constant if $d > n$,*
(b) *either constant or f has at least logarithmic growth if $d < n$, and*
(c) *either constant or f is not slow if $d = n$.*

This result yields the aforementioned Varopoulos's theorem as a corollary. Indeed, if the fundamental group of $\mathbb{Y}$ has order of growth at least $d > n$, then universal cover supports an d-dimensional isoperimetric inequality; we refer to [8, Theorem 6.19] for more details. Since a lift $\mathbb{R}^n \to \tilde{\mathbb{Y}}$ of a K-quasiregular mapping $\mathbb{R}^n \to \mathbb{Y}$ to the universal cover $\tilde{\mathbb{Y}}$ of $\mathbb{Y}$ is K-quasiregular, Varopoulos's theorem for quasiregular mappings now follows from (a) in Theorem 2.

Examples of open Riemannian manifolds supporting an isoperimetric inequality include nilpotent Lie groups and hyperbolic spaces. In many examples discussed in the literature, the isoperimetric dimension exceeds the topological dimension of the space. In such cases Theorem 2 has the role of a Picard theorem (case (a)). We refer to [19, Chapter IV], [5], and [8, Chapter 6] for these examples and a detailed discussion on isoperimetric inequalities, and [3] (and [4]) for Liouville and Picard theorems for quasiregular mappings under isoperimetric assumptions on the target space.

The original proof of Bonk and Heinonen is based on the non-linear Hodge theory and the $\mathcal{A}$-harmonic potential theory. In contrast, the proof of Varopoulos's theorem is based on an isoperimetric inequality and uses the fact that a closed manifold admitting a non-constant quasiregular mapping from $\mathbb{R}^n$ does not have a conformally hyperbolic universal cover. It seems that the effective use of the non-linear potential theory is tied to the uniform boundedness of the distortion. In our setting, we have found that more direct methods, that are purely analytic and use no conformal geometry, are easier to apply. Indeed, the proofs of Theorems 1 and 2 are based on the existence of integral type isoperimetric inequalities for suitable Sobolev mappings, and an interplay between volume growth and distortion.

The method of the proof of Theorem 1 relies on representation of the volume form of the target manifold using the Hodge theory and the Poincaré duality. Here our debt to the discussion in [9] on so-called *Cartan forms* is apparent. This method has the advantage that the obtained isoperimetric inequalities allow us to assume that the mean distortion of the mapping is in L^{n-1}. In the Euclidean theory of mappings of finite distortion, the L^{n-1}-integrability assumption on the distortion function is considered to be the minimal requirement for topological conclusions. For instance, the full analogue of Reshetnyak's theorem is conjectured to hold if $K \in L^{n-1}_{\mathrm{loc}}$ and known under a slightly stronger integrable assumption on K; that is, a non-constant mapping of finite distortion is both discrete and open provided

$K \in L^p_{\mathrm{loc}}$, $p > n-1$; see [14]. Apart from continuity, our techniques do not rely on topological properties of the mappings.

In the case of Theorem 2, we obtain an isoperimetric inequality for mappings through Gagliardo-Nirenberg-Sobolev type inequalities for BV functions. For Euclidean targets, this isoperimetric inequality is classical, see e.g. [17, p. 81].

The boundedness of mean L^p-distortion can be relaxed in both theorems. In Section 5 we prove sharp versions of Theorems 1 and 2 in terms of a gauge function defined in (5.10).

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2. Preliminaries

Throughout the article we consider continuous Sobolev mappings of the class $W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{Y})$, where $\mathbb{Y}$ is a connected and oriented Riemannian n-manifold without boundary. We define Sobolev spaces as in [9]. We fix a smooth Nash embedding $\iota\colon \mathbb{Y} \to \mathbb{R}^N$, and say that $f \in W^{1,n}(\Omega, \mathbb{Y})$ if coordinates of $\iota \circ f$ are Sobolev functions in $W^{1,n}(\Omega)$ where Ω is a domain in $\mathbb{R}^n$. We define the local space $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{Y})$ similarly.

We refer to [9] for a detailed treatment of Sobolev spaces of mappings between manifolds.

A Sobolev mapping $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{Y})$ induces a pull-back homomorphism $f^*\colon C^\infty(\bigwedge^\ell \mathbb{Y}) \to L^{n/\ell}_{\mathrm{loc}}(\mathbb{R}^n)$ locally of the form

$$f^*(u dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}) = (u \circ f) d(x_{i_1} \circ f) \wedge \cdots \wedge d(x_{i_\ell} \circ f).$$

Moreover, the pull-back f^* commutes with the exterior derivative, that is, $d \circ f^* = f^* \circ d$, where the left hand side is understood in the weak sense. By the pointwise inequality, $|f^*\alpha| \leqslant |Df|^\ell(|\alpha| \circ f)$ for $\alpha \in C^\infty(\bigwedge^\ell \mathbb{Y})$, we can conclude that $f^*\alpha \in W^{d,n/\ell}_{\mathrm{loc}}(\bigwedge^\ell \mathbb{R}^n)$ for all closed forms $\alpha \in C^\infty(\bigwedge^\ell \mathbb{Y})$. Here $W^{d,p}_{\mathrm{loc}}(\bigwedge^\ell \mathbb{R}^n)$ is the local partial Sobolev space of ℓ-forms, see [12] for more details. In particular, $f^*\alpha$ is weakly exact for every closed form $\alpha \in C^\infty(\bigwedge^\ell \mathbb{Y})$, since the Sobolev-de Rham cohomology $H^{\ell,n/\ell}(\mathbb{R}^n)$ is naturally isomorphic to the de Rham cohomology of $\mathbb{R}^n$; see e.g. [20, Chapter 5] for a standard sheaf argument and [16] for details in this particular case.

To define the *maximal splitting defect of a Riemannian metric*, suppose that $\mathbb{Y}$ is not a rational homology sphere and let g be a Riemannian metric on $\mathbb{Y}$.

Let vol_g be the volume form of metric g. By the Hodge theory and the Poincaré duality, for every non-trivial cohomology class $c \in H^\ell(\mathbb{Y})$ the volume form vol_g can be represented using the harmonic ℓ-form $\xi \in C^\infty(\bigwedge^\ell \mathbb{Y})$ in c as

$$\mathrm{vol}_g = \left(\frac{1}{\sqrt{\lambda}}\xi\right) \wedge * \left(\frac{1}{\sqrt{\lambda}}\xi\right) + d\tau \tag{2.5}$$

for some $(n-1)$-form τ, where

$$\lambda = \int_{\mathbb{Y}} \xi \wedge *\xi > 0.$$

If $\lambda = 1$, we say that the harmonic form ξ *almost splits* vol_g. If, in addition, $d\tau = 0$ in (2.5), we say that ξ *splits* vol_g. To measure the defect in splitting of the volume form of metric g, we denote for every harmonic ℓ-form ξ almost splitting vol_g *the splitting defect* of ξ by

$$C_g(\xi) = \inf_\tau (\|\xi\|_\infty^2 + \|\tau\|_\infty), \tag{2.6}$$

where the infimum is taken over all $(n-1)$-forms τ satisfying (2.5). Furthermore, we say that *the maximal splitting defect of* g is

$$C_g = \sup_\xi C_g(\xi), \tag{2.7}$$

where the supremum is taken over all non-trivial harmonic forms. The maximal splitting defect of g is finite by finite dimensionality of $H^*(\mathbb{Y})$.

3. Isoperimetric inequality for Sobolev mappings

In this section, we show that an isoperimetric inequality of the target space yields an integral type isoperimetric inequality for continuous Sobolev mappings in $W^{1,n}$.

We say that $\mathbb{Y}$ supports a d-dimensional isoperimetric inequality with a constant $C_{\mathbb{Y}} > 0$ if $\mathbb{Y}$ satisfies $(1,\varphi)$-isoperimetric inequality with $C_{\mathbb{Y}} > 0$, that is,

$$\varphi(|E|)|E| \leqslant C_{\mathbb{Y}}|\partial E|$$

for all compact sets $E \subset \mathbb{Y}$, where

$$\varphi(r) = \begin{cases} r^{-1/d}, & r > 1 \\ r^{-1/n}, & r \leqslant 1. \end{cases}$$

In particular, closed manifolds do not support an isoperimetric inequality and a manifold $\mathbb{Y}$ supporting a d-dimensional isoperimetric inequality supports also a d'-dimensional isoperimetric inequality for all $d' < d$, since $r^{-1/d'} < r^{-1/d}$ for $r > 1$.

Theorem 3. *Let $n \geqslant 2$ and $d \geqslant 2$. Let $\mathbb{Y}$ be a connected and oriented Riemannian n-manifold supporting a d-dimensional isoperimetric inequality with a constant $C_{\mathbb{Y}} > 0$. Let also $f\colon \mathbb{R}^n \to \mathbb{Y}$ be a continuous Sobolev mapping in $W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{Y})$. Then there exists $C = C(C_{\mathbb{Y}}, d) > 0$ so that for almost every $r > 0$ we have*

(i)

$$\int_{B^n(r)} J_f \leqslant C \left(\int_{\partial B^n(r)} |D^{\#} f| \right)^{\frac{d}{d-1}} \quad \text{if } d \geqslant n, \text{ and}$$

(ii)

$$\int_{B^n(r)} J_f \leqslant C \max \left\{ \mathcal{J}(r)^{1/n}, \mathcal{J}(r)^{1/d} \right\} \left(\int_{\partial B^n(r)} |D^{\#} f| \right)$$

if $d < n$; where $\mathcal{J}(r) = \int_{B^n(r)} J_f$.

Although the following lemma is certainly well-known, we include a proof for reader's convenience.

Lemma 4. *Let $\Omega \subset \mathbb{R}^n$ be an open set with a smooth boundary, $\mathbb{Y}$ an oriented Riemannian n-manifold, and $f \colon \mathbb{R}^n \to \mathbb{Y}$ a smooth mapping. Then*

$$y \mapsto \deg(y, \Omega; f)$$

is a BV function on $\mathbb{Y}$ and

$$|D \deg(\cdot, \Omega; f)|(\mathbb{Y}) \leqslant \int_{\partial\Omega} |D^{\#} f|.$$

An L^1-function $u \colon \mathbb{Y} \to \mathbb{R}$ is a said to be BV if

$$|Du|(\mathbb{Y}) := \sup \left\{ \int_{\mathbb{Y}} u \operatorname{div}\varphi \operatorname{vol}_{\mathbb{Y}} \colon \varphi \in C_0^\infty(T\mathbb{Y}),\ |\varphi| \leqslant 1 \right\} < \infty.$$

Proof of Lemma 4. Let φ be a compactly supported smooth vector field on $\mathbb{Y}$. Then, by definition,

$$\operatorname{div}\varphi \operatorname{vol}_{\mathbb{Y}} = d(\varphi \llcorner \operatorname{vol}_{\mathbb{Y}})$$

and by Stokes' theorem,

$$\int_\Omega df^*(\varphi \llcorner \operatorname{vol}_{\mathbb{Y}}) = \int_{\partial\Omega} \iota^* f^*(\varphi \llcorner \operatorname{vol}_{\mathbb{Y}}).$$

where $\varphi \llcorner \operatorname{vol}_{\mathbb{Y}}$ is the *contraction of* $\operatorname{vol}_{\mathbb{Y}}$ *by* φ, that is, $\varphi \llcorner \operatorname{vol}_{\mathbb{Y}}$ is the $(n-1)$-form defined by $(\varphi \llcorner \operatorname{vol}_{\mathbb{Y}})_y(v_1, \ldots, v_{n-1}) = (\operatorname{vol}_{\mathbb{Y}})_y(\varphi(y), v_1, \ldots, v_{n-1})$ for $y \in \mathbb{Y}$ and $v_1, \ldots, v_{n-1} \in T_y\mathbb{Y}$.

Thus

$$\begin{aligned}
\int_\Omega (\operatorname{div}\varphi) \circ f J_f \operatorname{vol}_{\mathbb{X}} &= \int_\Omega f^*(\operatorname{div}\varphi \operatorname{vol}_{\mathbb{Y}}) = \int_\Omega f^*(d(\varphi \llcorner \operatorname{vol}_{\mathbb{Y}})) \\
&= \int_\Omega df^*(\varphi \llcorner \operatorname{vol}_{\mathbb{Y}}) = \int_{\partial\Omega} \iota^* f^*(\varphi \llcorner \operatorname{vol}_{\mathbb{Y}}) \\
&\leqslant \int_{\partial\Omega} |D^{\#} f| |\varphi \llcorner \operatorname{vol}_{\mathbb{Y}}| \circ f \leqslant \|\varphi\|_\infty \int_{\partial\Omega} |D^{\#} f|.
\end{aligned}$$

By the change of variables,

$$\left| \int_{\mathbb{Y}} (\operatorname{div}\varphi) \deg(\cdot, \Omega; f) \operatorname{vol}_{\mathbb{Y}} \right| = \left| \int_\Omega (\operatorname{div}\varphi) \circ f J_f \operatorname{vol}_{\mathbb{X}} \right| \leqslant \|\varphi\|_\infty \int_{\partial\Omega} |D^{\#} f|.$$

This concludes the proof. ☐

The following L^1 estimate for BV functions is a combination of results of Coulhon, Grigoryán, and Levin [2, Prop 2.1] and Miranda, Pallara, Paronetto, and Preunkert [15, Prop 1.4].

Lemma 5. *Let $\mathbb{Y}$ support a d-dimensional isoperimetric inequality, $2 \leqslant d \leqslant n$, with constant $C_{\mathbb{Y}} > 0$ and let $u\colon \mathbb{Y} \to \mathbb{Z}$ be a compactly supported BV function on $\mathbb{Y}$. Then*

$$\|u\|_1 \leqslant C \max\{\|u\|_1^{1/d}, \|u\|_1^{1/n}\} |Du|(\mathbb{Y}),$$

where $C = C(C_{\mathbb{Y}}) > 0$

Proof. By considering $\tilde{\varphi} = \varphi / C_{\mathbb{Y}}$ if necessary, we may assume that $C_{\mathbb{Y}} = 1$. Since $\mathbb{Y}$ supports d-dimensional isoperimetric inequality, by [2, Prop. 2.1] it also supports F-Sobolev inequality for compactly supported Lipschitz functions, that is,

$$\int_{\mathbb{Y}} |v| F\left(\frac{v}{\|v\|_1}\right) \leqslant \int_{\mathbb{Y}} |\nabla v|,$$

where $F(r) = c\varphi(2/r)$, for every $v \in \mathrm{Lip}_0(\mathbb{Y})$. Here $c > 0$ is universal.

Suppose now that $u \in BV(\mathbb{Y})$ is compactly supported and let Ω be a relatively compact domain of $\mathbb{Y}$ containing the support of u. By [15, Prop. 1.4], there exists a sequence (u_k) in $C_0^\infty(\Omega)$ so that $u_k \to u$ in L^1 and

$$|Du|(\mathbb{Y}) = \lim_{k\to\infty} \int_{\mathbb{Y}} |\nabla u_k|.$$

Thus, by continuity of F and Fatou's lemma, we have

$$\begin{aligned} \int_{\mathbb{Y}} |u| F\left(\frac{|u|}{\|u\|_1}\right) &\leqslant \liminf_{k\to\infty} \int_{\mathbb{Y}} |u_k| F\left(\frac{|u_k|}{\|u_k\|_1}\right) \\ &\leqslant \liminf_{k\to\infty} \int_{\Omega} |\nabla u_k| = |Du|(\mathbb{Y}). \end{aligned}$$

Let $\Omega' = \{y\colon |u(y)| \leqslant 2\|u\|_1\}$. Then

$$F\left(\frac{|u(y)|}{\|u\|_1}\right) = c\varphi\left(2\frac{\|u\|_1}{|u(y)|}\right) = \begin{cases} c\left(\frac{u(y)}{2\|u\|_1}\right)^{1/n}, & y \notin \Omega' \\ c\left(\frac{u(y)}{2\|u\|_1}\right)^{1/d}, & y \in \Omega' \end{cases}$$

Thus

$$\begin{aligned}
\|u\|_1 &= \int_{\mathbb{Y}} |u| = \int_{\Omega'} |u| + \int_{\mathbb{Y}\setminus\Omega'} |u| \\
&\leqslant \int_{\Omega'} |u|^{1+\frac{1}{d}} + \int_{\mathbb{Y}\setminus\Omega'} |u|^{1+\frac{1}{n}} \\
&\leqslant C\left(\int_{\Omega'} |u| F\left(\frac{|u|}{\|u\|_1}\right)\|u\|_1^{1/d} + \int_{\mathbb{Y}\setminus\Omega'} |u| F\left(\frac{|u|}{\|u\|_1}\right)\|u\|_1^{1/n}\right) \\
&\leqslant C \max\{\|u\|_1^{1/d}, \|u\|_1^{1/n}\} \int_{\mathbb{Y}} |u| F\left(\frac{|u|}{\|u\|_1}\right) \\
&\leqslant C \max\{\|u\|_1^{1/d}, \|u\|_1^{1/n}\} |Du|(\mathbb{Y}),
\end{aligned}$$

where $C > 0$ is universal. □

Proof of Theorem 3. Let $r > 0$. We construct first a sequence of smooth mappings $f_k\colon B^n(2r) \to \mathbb{Y}$ tending to $f|B^n(2r)$ in $W^{1,n}(B^n(2r), \mathbb{Y})$ as follows. Let V be a tubular neighborhood of $\iota(\mathbb{Y})$ in $\mathbb{R}^N$ and $\pi\colon V \to \mathbb{Y}$ a smooth projection. We fix $\varepsilon > 0$ so that the ε-neighborhood of $fB^n(2r)$ in $\mathbb{R}^N$ is contained in V. By a standard convolution argument, we now find a sequence $\tilde{f}_k\colon B^n(2r) \to V$ tending to $\iota \circ f$ in $W^{1,n}(B^n(2r), \mathbb{R}^N)$. Then the mappings $f_k = \iota^{-1} \circ \pi \circ \tilde{f}_k\colon B^n(2r) \to \mathbb{Y}$ are smooth mappings tending to $f|B^n(2r)$ in $W^{1,n}(B^n(2r), \mathbb{Y})$, since π is smooth, ι is the fixed Nash embedding, and $(\iota \circ f)(\bar{B}^n(2r))$ is compact. We refer to [9, Section 2.4] for more details.

Suppose now that $d \geqslant n$. Since $r^{\frac{d-1}{d}} \leqslant r^{\frac{n-1}{n}}$ for $r \leqslant 1$, by the d-dimensional isoperimetric inequality there exists $C = C(C_{\mathbb{Y}}) > 0$ so that

$$|E| \leqslant C|\partial E|^{\frac{d}{d-1}}$$

for all compact sets E in $\mathbb{Y}$. Thus $\mathbb{Y}$ supports the classical Gagliardo-Nirenberg-Sobolev inequality

$$\|u\|_{\frac{d}{d-1}} \leqslant |Du|(\mathbb{Y}).$$

for BV functions u on $\mathbb{Y}$, see e.g. [10, 3.30] and [6, 5.6.2]. Thus, by Lemma 4, we have

$$\|\deg(\cdot, B^n(r), f_k)\|_{\frac{d}{d-1}} \leqslant C_{\mathbb{Y}} |D \deg(\cdot, B^n(r), f_k)|(\mathbb{Y}) \leqslant C_{\mathbb{Y}} \int_{\partial B^n(r)} |D^{\#} f_k|$$

for almost every $r > 0$ and every k. Since $\deg(\cdot, B^n(r), f_k)$ is integer valued,

$$\begin{aligned}
\int_{B^n(r)} J_{f_k} &= \int_{\mathbb{Y}} \deg(\cdot, B^n(r), f_k) \leqslant \int_{\mathbb{Y}} |\deg(\cdot, B^n(r), f_k)|^{\frac{d}{d-1}} \\
&\leqslant \left(C_{\mathbb{Y}} \int_{\partial B^n(r)} |D^{\#} f_k|\right)^{\frac{d}{d-1}}
\end{aligned}$$

for almost every $r > 0$.

Suppose now that $\mathbb{Y}$ supports a d-dimensional isoperimetric inequality for $2 \leqslant d < n$ with a constant $C_{\mathbb{Y}}$. By Lemma 5 and Lemma 4 together with the change of variables, we obtain

$$\int_{B^n(r)} J_{f_k} \leqslant C \max\left\{ \left(\int_{B^n(r)} J_{f_k} \right)^{1/n}, \left(\int_{B^n(r)} J_{f_k} \right)^{1/d} \right\} \left(\int_{\partial B^n(r)} |Df_k|^{n-1} \right).$$

for every $r > 0$ and every k. Thus for every $d \geqslant 2$, the claim holds for smooth mappings f_k.

By a usual telescope decomposition of the Jacobian in local coordinates [11, 8.1], we obtain

$$\int_{B^n(r)} J_{f_k} \to \int_{B^n(r)} J_f$$

as $k \to \infty$.

Using the telescope decomposition again, we have

$$\begin{aligned} \int_0^r \int_{\partial B^n(t)} \Big| |D^\# f_k| - |D^\# f| \Big| \, \mathrm{d}t &= \int_{B^n(r)} \Big| |D^\# f_k| - |D^\# f| \Big| \\ &\leqslant \int_{B^n(r)} \Big| D^\# f_k - D^\# f \Big| \to 0 \end{aligned}$$

for every $r > 0$ as $k \to \infty$. Thus

$$\int_{\partial B^n(r)} |D^\# f_k| \to \int_{\partial B^n(r)} |D^\# f|$$

for almost every $r > 0$. The claim follows. □

4. Cohomological isoperimetric inequality for Sobolev mappings

In this section we study mappings into closed target manifolds. We prove that the non-trivial kernel of the induced pull-back mapping yields an integral type isoperimetric inequality for mappings. This extends the result of Giannetti and Passarelli di Napoli [7].

Theorem 6. *Let $n \geqslant 2$ and let $(\mathbb{Y}, g_{\mathbb{Y}})$ be a closed, connected, and oriented Riemannian n-manifold, and $f\colon \mathbb{R}^n \to \mathbb{Y}$ a continuous Sobolev mapping in $W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{Y})$. Let $p \in [n-1, n)$. If either*

(i) *$p > n-1$ and there exists $\ell \in \{1, \dots, n-1\}$ such that $H^\ell(\mathbb{Y}) \neq 0$, or*

(ii) *$p = n-1$, $n \geqslant 4$, and there exists $\ell \in \{2, \dots, n-2\}$ such that $H^\ell(\mathbb{Y}) \neq 0$,*

then

$$\int_{B^n(r)} J_f \leqslant C |\partial B^n(r)|^{\frac{n}{n-1} - \frac{n}{p}} \left(\int_{\partial B^n(r)} |Df|^p \right)^{\frac{n}{p}} + C \int_{\partial B^n(r)} |Df|^{n-1} \tag{4.8}$$

for almost every $r > 0$, where $C = C(n, p, C_{g_{\mathbb{Y}}})$.

For the proof we recall the following version of Stokes' theorem.

Lemma 7. *Let $p > 1$ and $\omega \in W^{d,p}_{\mathrm{loc}}(\bigwedge^{n-1}\mathbb{R}^n)$. Then $\iota^*\omega \in W^{d,p}\left(\bigwedge^{n-1}\partial B^n(r)\right)$ and*

$$\int_{\partial B^n(r)} \iota^*\omega = \int_{B^n(r)} d\omega.$$

for almost every $r > 0$. Furthermore, if for $r > 0$ holds that $\omega \in W^{d,p}(\bigwedge^{n-1}\bar{B}^n(r))$ is weakly closed and $\iota^*\omega \in W^{d,p}(\bigwedge^{n-1}\partial B^n(r))$ then

$$\int_{\partial B^n(r)} \iota^*\omega = 0.$$

Proof. By the density of smooth forms, we may fix a sequence (ω_k) such that $\omega_k \to \omega$ in $W^{d,p}_{\mathrm{loc}}(\bigwedge^\ell \mathbb{R}^n)$. Since

$$\int_0^r \int_{\partial B^n(t)} |\iota^*(\omega-\omega_k)|^p \,\mathrm{d}t = \int_{B^n(r)} |\iota^*(\omega-\omega_k)|^p \leqslant \int_{B^n(r)} |\omega-\omega_k|^p \to 0$$

as $k \to \infty$, we have that

$$\int_{\partial B^n(r)} |\iota^*(\omega-\omega_k)|^p \to 0$$

for almost every $r > 0$. Hence, for almost every $r > 0$,

$$\int_{B^n(r)} d\omega = \lim_{k\to\infty} \int_{B^n(r)} d\omega_k = \lim_{k\to\infty} \int_{\partial B^n(r)} \iota^*\omega_k = \int_{\partial B^n(r)} \iota^*\omega.$$

Suppose now that $r > 0$ is such that $\omega \in W^{d,p}(\bigwedge^{n-1}\bar{B}^n(r))$ is weakly closed, and $\iota^*\omega \in W^{d,p}(\bigwedge^{n-1}\partial B^n(r))$. Then $\iota^*\omega$ is weakly closed and there exist sequences (η_k) and (η'_k) of smooth closed forms in $C^\infty(\bigwedge^{n-1}\bar{B}^n(r))$ and $C^\infty(\bigwedge^{n-1}U)$, where $U = B^n(r+\varepsilon)\setminus \bar{B}^n(r-\varepsilon)$ is a neighborhood of $\partial B^n(r)$, $\varepsilon > 0$, such that $\eta_k \to \omega$ and $\iota^*\eta'_k \to \iota^*\omega_k$ in $W^{d,p}(\bigwedge^{n-1}\bar{B}^n(r))$ and $W^{d,p}(\bigwedge^{n-1}\partial B^n(r))$, respectively. We may assume that $\eta'_k - \eta_k = d\tau_k$ in $U \cap B^n(r)$ for some smooth $(n-2)$-forms τ_k defined in U. Thus we may set $\omega_k = \eta_k + d(\varphi\tau_k)$, where $\varphi \in C^\infty_0(U)$ satisfies $\varphi \equiv 1$ in a neighborhood of $\partial B^n(r)$. Thus $\iota^*\omega_k = \iota^*\eta'_k$ and $d\omega_k = d\eta_k$ in $B^n(r)$. Then

$$\int_{\partial B^n(r)} \iota^*\omega = \lim_{k\to\infty} \int_{\partial B^n(r)} \iota^*\omega_k = \lim_{k\to\infty} \int_{B^n(r)} d\omega_k = 0.$$

The proof is complete. □

Proof of Theorem 6. Under either of the assumptions (i) and (ii), we may fix an harmonic ℓ-form ξ on $\mathbb{Y}$ such that $f^*\xi$ is weakly exact. Then

$$\mathrm{vol}_{\mathbb{Y}} = \xi \wedge *\xi + d\tau,$$

where $\tau \in C^\infty(\bigwedge^{n-1}\mathbb{Y})$. Thus

$$\int_{B^n(r)} J_f \mathrm{vol}_{\mathbb{R}^n} = \int_{B^n(r)} f^*(\mathrm{vol}_{\mathbb{Y}}) = \int_{B^n(r)} f^*(\xi)\wedge f^*(*\xi) + \int_{B^n(r)} f^*(d\tau)$$

for every $r > 0$. Since

$$\int_{B^n(r)} f^*(d\tau) = \int_{B^n(r)} df^*\tau = \int_{\partial B^n(r)} \iota^* f^*\tau \leqslant \|\tau\|_\infty \int_{\partial B^n(r)} |Df|^{n-1}$$

for almost every $r > 0$ by Stokes' theorem (Lemma 7), it suffices to show that

$$\int_{B^n(r)} f^*(\xi) \wedge f^*(*\xi) \leqslant C\|\xi\|_\infty^2 |\partial B^n(r)|^{\frac{n}{n-1}-\frac{n}{p}} \left(\int_{\partial B^n(r)} |Df|^p \right)^{\frac{n}{p}}$$

for almost every $r > 0$.

Suppose first that (i) holds. Since $f^*(\xi)$ is weakly exact, we may fix $\omega \in W^{d,n/\ell}_{\mathrm{loc}}(\bigwedge^{\ell-1}\mathbb{R}^n)$ so that $d\omega = f^*(\xi)$. Thus, by weak exactness of $f^*(*\xi)$,

$$\int_{B^n(r)} f^*(\xi) \wedge f^*(*\xi) = \int_{B^n(r)} d\omega \wedge f^*(*\xi) = \int_{B^n(r)} d(\omega \wedge f^*(*\xi))$$

for every $r > 0$.

We set

$$q = \frac{p}{\ell},\ q^* = \frac{(n-1)q}{(n-1)-q},\ \text{and } s = \frac{q^*}{q^*-1}.$$

For almost every $r > 0$ we may fix, by the Poincaré inequality for differential forms [12, Theorem 6.4], a weakly closed form $\tilde{\omega} \in W^{d,q}(\bigwedge^\ell B^n(r))$ such that $\iota^*\tilde{\omega} \in W^{d,q^*}\left(\bigwedge^{\ell-1}\partial B^n(r)\right)$ and

(4.9)
$$\left(\int_{\partial B^n(r)} |\iota^*\omega - \iota^*\tilde{\omega}|^{q^*} \,\mathrm{d}\mathcal{H}^{n-1} \right)^{1/q^*} \leqslant C_P \left(\int_{\partial B^n(r)} |d\iota^*\omega|^q \,\mathrm{d}\mathcal{H}^{n-1} \right)^{1/q}.$$

where C_P depends only on n. Let $\hat{\omega} = \omega - \tilde{\omega}$. Since $\tilde{\omega}$ is closed, we have, by Stokes' theorem,

$$\int_{\partial B^n(r)} \iota^*\tilde{\omega} \wedge \iota^* f^*(*\xi) = \int_{B^n(r)} d\tilde{\omega} \wedge f^*(*\xi) = 0.$$

Thus

$$\begin{aligned}\int_{B^n(r)} f^*(\xi) \wedge f^*(*\xi) &= \int_{\partial B^n(r)} \iota^*(\omega \wedge f^*(*\xi)) = \int_{\partial B^n(r)} \iota^*\hat{\omega} \wedge \iota^* f^*(*\xi) \\ &\leqslant C \int_{\partial B^n(r)} |\iota^*\hat{\omega}||\iota^* f^*(*\xi)| \leqslant C \int_{\partial B^n(r)} |\iota^*\hat{\omega}||f^*(*\xi)|,\end{aligned}$$

where C depends only on n. Since

$$|f^*(\xi)|^q \leqslant |Df|^{q\ell}(|\xi|^q \circ f) \leqslant \|\xi\|_\infty^q |Df|^p$$

and

$$|f^*(*\xi)|^s \leqslant |Df|^{s(n-\ell)} \|*\xi\|_\infty^s = \|\xi\|_\infty^s |Df|^{s(n-\ell)},$$

where

$$s(n-\ell) = p\frac{(n-1)(n-\ell)}{np-n\ell+\ell} < p\frac{(n-1)(n-\ell)}{n(n-1)-n\ell+\ell} = p,$$

we have, by Hölder's inequality and (4.9), that

$$\begin{aligned}
\int_{\partial B^n(r)} |\iota^*\hat{\omega}||f^*(*\xi)| &\leqslant \left(\int_{\partial B^n(r)} |\iota^*\hat{\omega}|^{q^*}\right)^{1/q^*} \left(\int_{\partial B^n(r)} |f^*(*\xi)|^s\right)^{1/s} \\
&\leqslant C_P \left(\int_{\partial B^n(r)} |d\hat{\omega}|^q\right)^{1/q} \left(\int_{\partial B^n(r)} |f^*(*\xi)|^s\right)^{1/s} \\
&\leqslant C_P \left(\int_{\partial B^n(r)} |f^*(\xi)|^q\right)^{1/q} \left(\int_{\partial B^n(r)} |f^*(*\xi)|^s\right)^{1/s} \\
&\leqslant C_P |\partial B^n(r)|^{\frac{n}{n-1}-\frac{n}{p}} \|\xi\|_\infty^2 \left(\int_{\partial B^n(r)} |Df|^p\right)^{\frac{n}{p}}.
\end{aligned}$$

This proves (4.8).

We assume now (ii). Set

$$q_0 = \frac{n-1}{\ell},\ q_0^* = \frac{(n-1)q_0}{(n-1)-q_0},\ \text{and } s_0 = \frac{q_0^*}{q_0^*-1}.$$

Then $s_0(n-\ell) = n-1$. Following the proof above by almost verbatim, we have

$$\begin{aligned}
\int_{B^n(r)} f^*(\xi)\wedge f^*(*\xi) &\leqslant C_P \left(\int_{\partial B^n(r)} |f^*(\xi)|^{q_0}\right)^{1/q_0} \left(\int_{\partial B^n(r)} |f^*(*\xi)|^{s_0}\right)^{1/s_0} \\
&\leqslant C_P \|\xi\|_\infty^2 \left(\int_{\partial B^n(r)} |Df|^{n-1}\right)^{\frac{n}{n-1}}.
\end{aligned}$$

This concludes the proof. □

5. Growth of mappings of finite distortion

In this section we state and prove sharp versions of Theorems 1 and 2. First, we consider Theorem 2 in terms of a growth condition of the Jacobian. For this, let $K\colon \mathbb{R}^n \to [1,\infty)$ be a measurable function, and $p \geqslant 1$. We set

$$\mathfrak{K}_p(r) = \mathfrak{K}_{K,p}(r) = \left(\fint_{\partial B^n(r)} K^p\right)^{1/p}$$

for $r > 0$ whenever the integral on the right is defined. We define also $\psi_p\colon (0,\infty) \to (0,\infty)$ by

$$\psi_p(r) = \psi_{\mathfrak{K}_p}(r) = \int_1^r \frac{\mathrm{d}s}{s\mathfrak{K}_p(s)}. \tag{5.10}$$

Theorem 8. *Let $\mathbb{Y}$ be a connected and oriented Riemannian n-manifold supporting a d-dimensional isoperimetric inequality, $d \geqslant 2$, with a constant $C_{\mathbb{Y}} > 0$. Let $f\colon \mathbb{R}^n \to \mathbb{Y}$ be a mapping of finite distortion with distortion function $K\colon \mathbb{R}^n \to \mathbb{R}$. Suppose that $\psi_{\mathfrak{K}_{n-1}}(r) \to \infty$ as $r \to \infty$. Then f is constant if $d > n$.*

Furthermore, if $d \leqslant n$, then either f is constant or we have the following two cases:

(a) *For $d < n$ there exists $\alpha = \alpha(n, d) > 0$ such that*
$$\liminf_{r\to\infty} \frac{1}{(\psi_{n-1}(r))^{\alpha}} \int_{B^n(r)} J_f > 0.$$
(b) *For $d = n$ there exists $\beta = \beta(n, C_{\mathbb{Y}}) > 0$ such that*
$$\liminf_{r\to\infty} \frac{1}{e^{\beta\psi_{n-1}(r)}} \int_{B^n(r)} J_f > 0.$$

Proof. Suppose first that $d \geqslant n$. By Theorem 3 and Hölder's inequality,
$$\begin{aligned}\int_{B^n(r)} J_f &\leqslant \left(C_{\mathbb{Y}} \int_{\partial B^n(r)} |D^{\#} f|\right)^{\frac{d}{d-1}} \\ &\leqslant C_{\mathbb{Y}}^{\frac{d}{d-1}} \left(\int_{\partial B^n(r)} K^{n-1}\right)^{\frac{1}{n}\frac{d}{d-1}} \left(\int_{\partial B^n(r)} J_f\right)^{\frac{n-1}{n}\frac{d}{d-1}} \\ &\leqslant C_{\mathbb{Y}}^{\frac{d}{d-1}} |\partial B^n(r)|^{\frac{1}{\gamma(n-1)}} \mathfrak{K}_{n-1}(t)^{1/\gamma} \left(\int_{\partial B^n(r)} J_f\right)^{1/\gamma}\end{aligned}$$
for almost every $r > 0$, where
$$\gamma = \frac{n}{n-1}\frac{d-1}{d}.$$
Set
$$\varphi(t) = \int_{B^n(r)} J_f$$
for $r > 0$. Then
$$\varphi(r)^{\gamma} \leqslant C_{\mathbb{Y}}^{\frac{n}{n-1}} |\partial B^n(r)|^{\frac{1}{n-1}} \mathfrak{K}_{n-1}(r)\varphi'(r) = C_{\mathbb{Y}}^{\frac{n}{n-1}} |\partial B^n|^{\frac{1}{n-1}} r\mathfrak{K}_{n-1}(r)\varphi'(r)$$
for almost every $r > 0$.

Suppose that f is non-constant. Since f is a mapping of finite distortion, there exists $r_0 > 0$ so that $\varphi(r_0) > 0$.

Then, for every $r \geqslant r_0$, we have that
$$\int_{r_0}^{r} \frac{\varphi'(s)}{\varphi(s)^{\gamma}}\,\mathrm{d}s \geqslant \beta \int_{r_0}^{r} \frac{\mathrm{d}s}{s\mathfrak{K}_{n-1}(s)} \geqslant \beta(\psi_{n-1}(r) - \psi_{n-1}(r_0)), \tag{5.11}$$
where $\beta = \left(C_{\mathbb{Y}}^{\frac{n}{n-1}} |\partial B^n|^{\frac{1}{n-1}}\right)^{-1}$.

If $d = n$ then $\gamma = 1$ and we obtain (b). Indeed, by (5.11),

$$\varphi(r) \geqslant \varphi(r_0) e^{\beta(\psi_{n-1}(r) - \psi_{n-1}(r_0))}$$

for $r \geqslant r_0$.

For $d > n$, we have $\gamma > 1$ and

$$\frac{\varphi(r)^{1-\gamma} - \varphi(r_0)^{1-\gamma}}{1-\gamma} \geqslant \beta(\psi_{n-1}(r) - \psi_{n-1}(r_0)).$$

If f is non-constant, there exists $r > r_0$ so that $\varphi(r) < 0$, which contradicts the positivity of the Jacobian of f. Thus f is constant for $d > n$.

Suppose now that $d < n$. Then, by Theorem 3,

$$\int_{B^n(r)} J_f \leqslant C \left\{ \left(\int_{B^n(r)} J_f \right)^{1/n}, \left(\int_{B^n(r)} J_f \right)^{1/d} \right\} \left(\int_{\partial B^n(r)} |Df|^{n-1} \right) \tag{5.12}$$

for almost every $r > 0$, where $C = C(C_{\mathbb{Y}}, d) > 0$. We show first that there exists $r_2 > 0$ so that

$$\int_{B^n(r)} J_f \leqslant C \left(\int_{\partial B^n(r)} |Df|^{n-1} \right)^{\frac{d}{d-1}}$$

for almost every $r \geqslant r_2$. It is enough to show that there exists $r > 0$ so that

$$\varphi(r) = \int_{B^n(r)} J_f > 1.$$

Suppose towards contradiction that $\varphi(r) \leqslant 1$ for all $r > 0$, then (5.12) yields

$$\int_{B^n(r)} J_f \leqslant C \left(\int_{\partial B^n(r)} |Df|^{n-1} \right)^{\frac{n}{n-1}}$$

for all $r > 0$. Following the argument of the case $d \geqslant n$, we obtain that

$$\varphi(r) \leqslant C_{\mathbb{Y}}^{\frac{n}{n-1}} |\partial B^n|^{\frac{1}{n-1}} r \mathfrak{K}_{n-1}(r) \varphi'(r)$$

for all $r > 0$ and that φ is not bounded. This is a contradiction and such $r_2 > 0$ exists.

We set

$$\gamma = \frac{n}{n-1} \frac{d-1}{d} < 1$$

as above. We may assume that $r_0 > r_2$.

Since (5.11) yields

$$\varphi(r)^{1-\gamma} \geqslant \varphi(r_0)^{1-\gamma} + (1-\gamma)\beta(\psi_{n-1}(r) - \psi_{n-1}(r_0)),$$

we have that (a) holds with $\alpha = 1/(1-\gamma)$ in the case $d < n$. $\square$

Proof of Theorem 2. Theorem 2 follows immediately from Theorem 8. Since

$$\mathcal{K}_{n-1} = \sup_{r>0} \left(\fint_{B^n(r)} K_f^{n-1}(x)\,\mathrm{d}x \right)^{1/(n-1)} < \infty$$

and

$$\begin{aligned} \fint_{r/2}^{r} \fint_{B^n(t)} K_f^{n-1} &\leqslant C \fint_{r/2}^{r} \frac{1}{t^{n-1}} \int_{B^n(t)} K_f^{n-1} \\ &\leqslant \frac{C}{r^n} \int_{B^n(r)} K_f^{n-1} \leqslant \mathcal{K}_{n-1}^{n-1}, \end{aligned}$$

where $C = C(n)$, we have that there exists a set $I \subset (0,\infty)$ so that $|I \cap (0,r)| \geqslant r/4$ and

$$\fint_{\partial B^n(r)} K_f^{n-1} \leqslant C\mathcal{K}_{n-1}^{n-1},$$

for $r \in I$, where $C = C(n)$. Thus $\mathfrak{K}_{n-1}(r) \leqslant C\mathcal{K}_{n-1}$ for $r \in I$ and

$$\begin{aligned} \psi_{n-1}(r) &= \int_1^r \frac{\mathrm{d}s}{s\mathfrak{K}_{n-1}(s)} \geqslant \frac{1}{C\mathcal{K}_{n-1}} \int_{I\cap(1,r)} \frac{\mathrm{d}s}{s} \\ &\geqslant \frac{1}{C\mathcal{K}_{n-1}} (\log r - \log 4) \geqslant \frac{1}{2C\mathcal{K}_{n-1}} \log r \end{aligned}$$

for $r > 16$, where $C = C(n)$. The claim now follows from Theorem 8 $\square$

Next we formulate a sharp version of Theorem 1.

Theorem 9. *Let $(\mathbb{Y}, g_{\mathbb{Y}})$ be a closed, connected, and oriented Riemannian n-manifold and let $f\colon \mathbb{R}^n \to \mathbb{Y}$ be a non-constant mapping of finite distortion with distortion $K\colon \mathbb{R}^n \to [1,\infty)$. Let $p \in [n-1, n)$ and that either*

(i) *$p > n-1$ and $\ker f^* \neq 0$, or*
(ii) *$p = n-1$, $n \geqslant 4$, and $H^\ell(\mathbb{Y}) \neq 0$ for some $\ell \in \{2,\dots,n-2\}$.*

If $\psi_{K,p}(r) \to \infty$ as $r \to \infty$ then there exists $\alpha = \alpha(n,p,C_{g_{\mathbb{Y}}}) > 0$ so that

$$\liminf_{r\to\infty} \frac{1}{e^{\alpha\psi_p(r)}} \int_{B^n(r)} J_f > 0. \tag{5.13}$$

Proof. We set $q = np/(p+1)$. Then $n-1 \leqslant q < n$ and $1 - n/q = 1/p$. Applying Theorem 6 with exponent q and Hölder's inequality with n/q and with $n/(n-1)$, we obtain, for $r \geqslant 1$,

(5.14)

$$\begin{aligned}
\int_{B^n(r)} J_f &\leqslant C|\partial B^n(r)|^{\frac{n}{n-1}-\frac{n}{q}} \left(\int_{\partial B^n(r)} |Df|^q \right)^{\frac{n}{q}} + C \int_{\partial B^n(r)} |Df|^{n-1} \\
&\leqslant C|\partial B^n(r)|^{\frac{n}{n-1}-\frac{n}{q}} \left(\int_{\partial B^n(r)} K^p \right)^{1/p} \int_{\partial B^n(r)} J_f \\
&\quad + C \left(\int_{\partial B^n(r)} K^{n-1} \right)^{1/n} \int_{\partial B^n(r)} J_f \\
&\leqslant C|\partial B^n(r)|^{\frac{1}{n-1}} \mathfrak{K}_p(r) \int_{\partial B^n(r)} J_f \\
&\quad + C|\partial B^n(r)|^{\frac{1}{n}} \mathfrak{K}_{n-1}(r)^{\frac{n-1}{n}} \int_{\partial B^n(r)} J_f \\
&\leqslant C|\partial B^n| r \mathfrak{K}_p(t) \int_{\partial B^n(r)} J_f,
\end{aligned}$$

where C is the constant in Theorem 6; we use here the observation that $\mathfrak{K}_p(r) \geqslant \mathfrak{K}_{n-1}(r) \geqslant \mathfrak{K}_{n-1}^{\frac{n-1}{n}}(r) \geqslant 1$ for almost every $r > 0$.

Let

$$\varphi(r) = \int_{B^n(r)} J_f$$

for $r > 0$. Then

$$\varphi'(r) = \int_{\partial B^n(r)} J_f$$

for almost every $r > 0$. Since f is not constant, there exists $r_\circ \geqslant 1$ so that $\varphi(r_\circ) > 0$. Thus

$$\log \frac{\varphi(r)}{\varphi(r_\circ)} \geqslant \int_{r_\circ}^{r} \frac{\varphi'(s)}{\varphi(s)}\,\mathrm{d}s \geqslant \alpha \int_{r_\circ}^{r} \frac{\mathrm{d}s}{s\mathfrak{K}_p(s)} \geqslant \alpha\left(\psi(r) - \psi(r_\circ)\right), \tag{5.15}$$

where α depends on n, p, and $C_{g_{\mathbb{Y}}}$. Thus

$$\frac{\varphi(r)}{e^{\alpha\psi(r)}} \geqslant \varphi(r_\circ) e^{-\alpha\psi(r_\circ)}$$

for $r \geqslant r_\circ$. The claim follows. □

Theorem 1 now follows from Theorem 9 using the argument in the proof of Theorem 2. We omit the details.

Example 10. *The following construction demonstrates the sharpness of the assumption $\psi_{K,p}(r) \to \infty$ as $r \to \infty$ in Theorems 8 and 9. The construction is well-known in the context of locally quasiregular mappings and due to Zorich* [21]*; see also* [13] *for a Liouville type theorem in this context. We thank Peter Lindqvist for these references.*

Suppose that we are given a measurable function $M\colon [1,\infty) \to [1,\infty)$ *so that*

$$\int_1^\infty \frac{\mathrm{d}t}{tM(t)} < \infty. \tag{5.16}$$

Then first, we define $\tilde{M}(t) = M(t)$ *for* $t \geqslant 1$ *and* $\tilde{M}(t) = t^{-2}$ *for* $0 < t < 1$. *Second, we set*

$$F(s) = \int_0^s \frac{\mathrm{d}t}{t\tilde{M}(t)} \qquad \text{and} \qquad \tilde{F}(s) = \frac{F(s)}{F(\infty)},$$

where $F(\infty) = \lim_{s\to\infty} F(s)$. *Now, we are ready to define a homeomorphism* $f\colon \mathbb{R}^n \to B^n$ *by the rule*

$$f(x) = \tilde{F}(|x|)\frac{x}{|x|}$$

for all $x \in \mathbb{R}^n$. *A simple computation shows that the mapping* f *has finite distortion* K *with*

$$\left(⨍_{\partial B^n(r)} K^p \right)^{1/p} \sim M(t)$$

for almost every $t \geqslant 1$ *and all* $p \geqslant n-1$, *where the constants of comparability depend only on* $F(\infty)$. *Together with the assumption* (5.16) *we have that* $\sup_{r\geqslant 1}\psi_p(r) < \infty$. *Since* B^n *can be embedded into any manifold with a bilipschitz embedding, our construction is completed.*

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