Quasiregular curves

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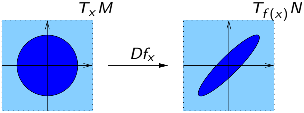
Definition

A continuous mapping $f\colon M \to N$ between oriented Riemannian n-manifolds is K-quasiregular if

- $f \in W^{1,n}_{\mathrm{loc}}(M,N)$, and
- $\|Df\|^n \le K \det Df$ almost everywhere.



In this terminology

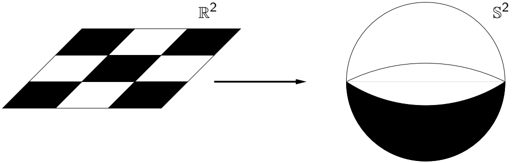
quasiconformal = quasiregular homeomorphism.

Examples

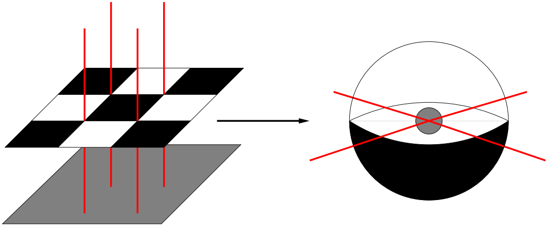
- holomorphic maps $\mathbb{C} \to \mathbb{C}$
- conformal maps $\mathbb{R}^n \to \mathbb{R}^n$
- PL branched covers $M \to N$ between closed manifolds.

Example: Zorich map $\mathbb{R}^3 \to \mathbb{R}^3 \setminus \{0\}$

Let $A\colon \mathbb{R}^2 \to \mathbb{S}^2$ be



Define $Z\colon \mathbb{R}^3 \to \mathbb{R}^3 \setminus \{0\}$ by $(x, y, t) \mapsto e^t A(x, y)$.



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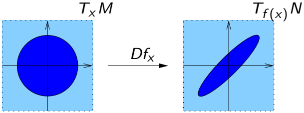

$n = 2$: Quasiregular maps between Riemann surfaces

- f is quasiregular $\Leftrightarrow$ $\partial_{\bar z} f = \mu \partial_z f$ (Beltrami equation).
- $f = (\text{holomorphic}) \circ (\text{quasiconformal})$ (Stoïlow's theorem)

$n \ge 3$:

- f quasiregular $\Leftrightarrow$ $J_f^{-2} (Df)^T Df = G$ (Beltrami equation)
- Reshetnyak: A non-constant qr-map is discrete, open, and $J_f > 0$ a.e.

Picard theorems

Theorem (Rickman's Picard theorem)

For a non-constant K-quasiregular map $f : \mathbb{R}^n \to \mathbb{S}^n$,

$$\#(\mathbb{S}^n \setminus f\mathbb{R}^n) \le C(n, K).$$

Theorem (Prywes's theorem)

If N is closed and there exists a non-constant quasiregular map $\mathbb{R}^n \to N$, then

$$\dim H^k(N) \le \dim H^k(\mathbb{T}^n).$$

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Methods: Elliptic PDE's of n-Laplace type

$$\Delta_n u = \mathrm{div}(|\nabla u|^{n-2} \nabla u) = 0 \Rightarrow \mathrm{div}(\mathcal{A}(u \circ f)) = 0.$$

Main question of the talk:

How to define quasiregularity if $\dim M < \dim N$?

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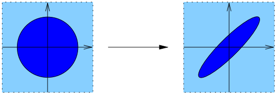
Motivating questions:

- Local topological results, like discreteness?
- Analytical results, like higher integrability?
- Picard type theorems?
- Connection to holomorphic curves?

Issues



Definition of quasiregularity uses Jacobian

Idea: Replace Jacobian condition with a ratio condition on semi-axes.

- Lose the sign of the Jacobian $\Rightarrow$ lose orientation.
- Low regularity $\Rightarrow$ folding is allowed (i.e. $(x, y) \mapsto (|x|, y)$).

Jacobian $\approx$ local degree theory

Local degree is given by generators of the cohomology in the top dimension.

- No local degree $\Rightarrow$ no Reshetnyak's theorem (?)

Idea: Compensate the lack of topology by an additional structure

Suppose $n = \dim M < \dim N = m$. An n-form $\omega \in \Omega^n(N)$ on N is an *n-volume form* if ω is closed and non-vanishing, that is, $\omega_p \neq 0$ for each $p \in N$.

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Let M be an n-dimensional oriented Riemannian manifold, let N be an m-dimensional Riemannian manifold for $m \geq n$.

Definition

A continuous mapping $f\colon M \to N$ is a K-quasiregular ω-curve with respect to an n-volume form $\omega \in \Omega^n(N)$ if

- $f \in W^{1,n}_{\mathrm{loc}}(M, N)$, and
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Here the norms are the operator norm and comass:

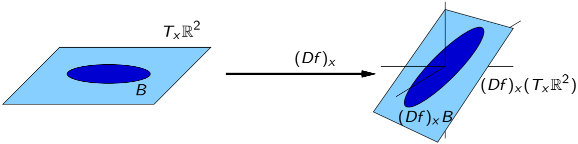
$$\|Df\|_x = \max\{|(Df)_x(v)| \colon |v| = 1\}$$

$$\|\omega\|_p = \max\{\omega_p(v_1, \ldots, v_n)\colon v_1, \ldots, v_n \in T_pN,\ \|v_j\| = 1\}$$

Geometric interpretation (an example)

Suppose

- ω is simple, e.g. $\omega = dx \wedge dy \in \bigwedge^2 \mathbb{R}^3$, and
- $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ a quasiregular ω-curve.



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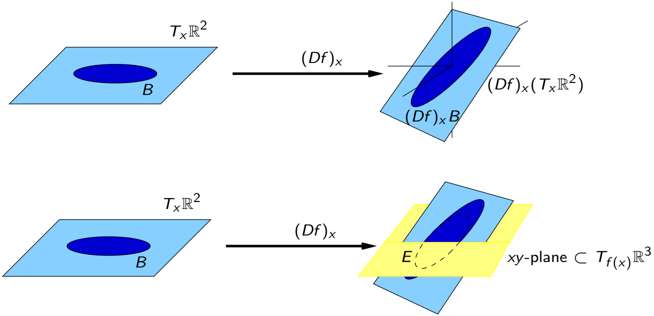

In this case:

Distortion K of f is given by the eccentricity of the ellipsoid E.

Examples

Quasiregular mappings are quasiregular curves

Suppose $m = n$ and take $\omega = \mathrm{vol}_N$. Then a K-quasiregular mapping $f\colon M \to N$ is a K-quasiregular ω-curve.

Holomorphic curves are 1-quasiregular curves

A holomorphic curve $h = (h_1, \ldots, h_k)\colon \mathbb{C} \to \mathbb{C}^k$ is a 1-quasiregular curve with respect to $\omega_0 = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$.
Reason: Since $\|\omega_0\| = 1$ and each Dh_j is conformal, we have

$$\|Dh\|^2 = \|Dh_1\|^2 + \cdots + \|Dh_k\|^2 = J_{h_1} + \cdots + J_{h_k} = h^*\omega_0.$$

Pseudoholomorphic curves are quasiregular curves (at least locally)

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Open question:

Does there exists a non-constant quasiregular curve $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ with respect to $\omega = \star\theta_H = \star(dt - \frac{1}{2}(xdy - ydx))$?

Properties of quasiregular curves

Theorem (Locally uniform limits are in the same class)

Let (f_k) be a sequence of K-quasiregular ω-curves $f_k \colon M \to N$ converging locally uniformly to a map $f \colon M \to N$. Then f is a K-quasiregular ω-curve.

Theorem (Liouville's theorem)

Let N be a complete Riemannian manifold and $\omega \in \Omega^n(N)$ an exact n-volume form. Then bounded quasiregular ω-curves $\mathbb{R}^n \to N$ are constant.

Method:

Both proofs are (partly) based on a simple Caccioppoli inequality: if $\omega = d\tau$ then, for $f \colon M \to N$ and $\psi \in C_0^\infty(M)$, we have

$$\int_M \psi^n f^*\omega \le C(n)K^{n-1} \int_M |\nabla \psi|^n \left(\frac{\|\tau\|^{\frac{n}{n-1}}}{\|\omega\|} \right) \circ f.$$

Liouville theorems for conformal curves

Theorem (Two dimensional 1-quasiregular curves are holomorphic)

Let $\Omega \subset \mathbb{R}^2$ be a domain and let $f = (f_1, \ldots, f_k) \colon \Omega \to (\mathbb{R}^2)^k$ be a 1-quasiregular curve with respect to the standard symplectic form $\omega_0 = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$. Then f is a holomorphic curve $\Omega \to \mathbb{C}^k$.

Theorem (Higher dimensional 1-quasiregular curves)

Let $\Omega \subset \mathbb{R}^n$ be a domain for $n \geq 3$ and let $f \colon \Omega \to (\mathbb{R}^n)^k$ be a 1-quasiregular ω-curve with respect to $\omega = \sum_{j=1}^s \pi_j^* \mathrm{vol}_{\mathbb{R}^n} \in \bigwedge^n \mathbb{R}^{nk}$. Then coordinate maps of $f = (f_1, \ldots, f_k) \colon \Omega \to (\mathbb{R}^n)^k$ are conformal.

Gromov's quasiminimality condition

Definition

A mapping $f\colon M \to N$ is C-quasiminimal if for each compact submanifold $W \Subset M$ with boundary and each competing map $h\colon M \to N$ we have

$$\int_W \|\wedge^n Df\| \le C \int_W \|\wedge^n Dh\|.$$

Theorem

A K-quasiregular ω-curve $f\colon M \to N$ is $K\mathrm{R}(\omega)$-quasiminimal, where

$$\mathrm{R}(\omega) = \frac{\max\|\omega\|}{\min\|\omega\|}.$$

Failure of Reshetnyak's theorem

Reshetnyak's theorem for quasiregular mappings:

A non-constant quasiregular mapping is discrete and open.

Theorem (Iwaniec–Verchota–Vogel, 2002)

There exists a Lipschitz map $F = (f_1, f_2) \colon \mathbb{C} \to \mathbb{C}^2$ having the following properties:

- $J_{f_1} + J_{f_2} \equiv 1$ *on* $\mathbb{C}_+$ *and*
- $F \equiv 0$ *on* $\mathbb{C}_-$.

The map $F = (f_1, f_2)$ in strip $\mathbb{R} \times (1/2, 1)$:

$$f_1(x,y) = \alpha(y) e^{-i2x} + \beta(y) e^{-ix}$$
$$f_2(x,y) = -\alpha(y) e^{-i2x} + \beta(y) e^{-ix}$$

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- $F \equiv 0$ *on* $\mathbb{C}_-$*.*

Consequence:

F is a quasiregular curve with respect to the symplectic form ω_0
$\Rightarrow$ Reshetnyak's theorem fails for quasiregular curves in general.

Curves over simple forms

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a domain and $f : \Omega \to \mathbb{R}^m$ be a quasiregular ω-curve with respect to a simple form $\omega \in \Omega^n(\mathbb{R}^m)$. Then, locally, f is a graph over a quasiregular mapping: for each $x \in \Omega$ there exists a neighborhood D, an isometry $L\colon \mathbb{R}^m \to \mathbb{R}^m$, and a quasiregular map $\hat{f}\colon D \to \mathbb{R}^n$ for which

$$L \circ f = (\hat{f}, u)\colon D \to \mathbb{R}^n \times \mathbb{R}^{m-n},$$

where $u \in W^{1,n}_{\mathrm{loc}}(D, \mathbb{R}^{m-n})$.

Corollary

- *f is discrete*
- *$\star f^*\omega > 0$ a.e.*
- *(higher integrability) exists $p > n$ for which $f \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$.*