MAPPINGS OF BOUNDED MEAN DISTORTION AND COHOMOLOGY

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Abstract. We obtain a quantitative cohomological boundedness theorem for closed manifolds receiving entire mappings of bounded mean distortion and finite lower order. We also prove an equidistribution theorem for mappings of finite distortion.

1. Introduction

By the classical *Uniformization Theorem*, the sphere $\mathbb{S}^2$ and the torus $\mathbb{T}^2$ are the only closed Riemann surfaces admitting nonconstant conformal mappings from the complex plane. The same rigidity is present in higher dimensions; closed manifolds admitting conformal mappings from $\mathbb{R}^n$ are quotients of $\mathbb{S}^n$ and $\mathbb{T}^n$, see e.g. [2, Prop. 1.4]. However, if distortion is allowed, simple examples show that the spaces $\mathbb{S}^{k_1} \times \mathbb{S}^{k_2} \times \cdots \times \mathbb{S}^{k_\ell}$ ($k_1 + \cdots + k_\ell = n$) receive nonconstant mappings of bounded distortion from $\mathbb{R}^n$. A mapping $f\colon M \to N$ between oriented Riemannian n-manifolds is said to be a *mapping of bounded distortion*, or *quasiregular*, if f is a Sobolev mapping in $W^{1,n}_{\mathrm{loc}}(M;N)$ and there exists a constant $K \geq 1$ so that

$$|Df(x)|^n \leq K J_f(x) \quad \text{for almost every } x \in M,$$

where $|Df(x)|$ is the operator norm of the differential $Df(x)$ and $J_f(x)$ is the Jacobian determinant of f at x. Mappings we consider are continuous and the Sobolev space $W^{1,n}_{\mathrm{loc}}(M;N)$ is understood as in [4]. By Reshetnyak's theorem [17, p. 163], quasiregular mappings are discrete and open, and therefore examples of generalized branched covers.

A connected and oriented Riemannian n-manifold receiving a nonconstant (K-)quasiregular mapping from $\mathbb{R}^n$ is called (K-)*quasiregularly elliptic*. By the Uniformization Theorem and the measurable Riemann Mapping Theorem, the only closed quasiregularly elliptic 2-manifolds are $\mathbb{S}^2$ and $\mathbb{T}^2$. For $n = 3$, closed quasiregularly elliptic manifolds are by Jormakka's theorem [10] quotients of $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{S}^1$, and $\mathbb{T}^3$. In higher dimensions such characterizations are not known. In dimension $n = 4$, a construction of Rickman [19]

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gives a positive answer to a question of Gromov [3, 2.41] on the quasiregular ellipticity of $\mathbb{S}^2 \times \mathbb{S}^2 \# \mathbb{S}^2 \times \mathbb{S}^2$.

The fundamental group and the de Rham cohomology ring yield obstructions for quasiregular ellipticity of a closed manifold. More precisely, by Varopoulos's theorem [23, Theorem X.11] the order of growth of the fundamental group of a closed quasiregularly elliptic manifold cannot exceed the dimension of the manifold. Similarly, by a theorem of Bonk and Heinonen [2, Theorem 1.1]: *Given $n \geq 2$ and $K \geq 1$ there exists a constant $C = C(n, K) > 0$ so that the dimension of the de Rham cohomology ring $H^*(N)$ of a closed K-quasiregularly elliptic n-manifold N is at most C.* Here, and in what follows, the dimension of $H^*(N)$ is $\dim H^*(N) = \sum_k \dim H^k(N)$.

Local versions of these theorems show that analogous results hold for mappings that are quasiregular in a neighborhood of infinity; see [15]. The assumption of quasiregularity can, however, be further relaxed. With Onninen we showed in [14] that Varopoulos's theorem holds for a larger class of mappings, a subclass of *mappings of finite distortion*. In this vein, we show that a cohomological boundedness phenomenon of Bonk-Heinonen type holds for a subclass of *mappings of bounded mean distortion*. To state our main results, we give some definitions.

We say that a nonconstant continuous mapping $f \colon \mathbb{R}^n \to N$ is a *mapping of finite distortion* if f belongs to the Sobolev space $W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n; N)$ and there exists a measurable function $K \colon \mathbb{R}^n \to [1, \infty)$ so that

$$|Df(x)|^n \leq K(x) J_f(x) \quad \text{for almost every } x \in \mathbb{R}^n.$$

We set the *outer distortion function* K_f of f to be the function $K_f(x) = |Df(x)|^n / J_f(x)$, whenever $J_f(x) > 0$, and $K_f(x) = 1$, otherwise.

We say that a mapping of finite distortion f has *($\mathcal{K}$-)bounded p-mean distortion*, $p \geq 1$, if there exist constants $\mathcal{K} \geq 1$ and $r_0 > 0$ so that

$$\left(\frac{1}{|B^n(r)|} \int_{B^n(r)} K_f^p(x) \, \mathrm{d}x \right)^{1/p} \leq \mathcal{K}$$

for every $r \geq r_0$. Here $B^n(r)$ is the open ball of radius r about the origin in $\mathbb{R}^n$ and $|B^n(r)|$ is the Lebesgue n-measure of $B^n(r)$. We also say that f has *finite lower order* λ if

$$\lambda = \liminf_{r \to \infty} \frac{\log A_f(r)}{\log r} < \infty.$$

Here and in what follows A_f is the *averaged counting function*

$$A_f(r) = \int_{B^n(r)} J_f(x) \, \mathrm{d}x.$$

Theorem 1. *For every $n \geq 2$, $\lambda \geq 0$, and $\mathcal{K} \geq 1$ there exist constants $p = p(n) > n - 1$ and $C = C(n, \lambda, \mathcal{K}) > 0$ with the following property. Let N be a closed, connected, and oriented Riemannian n-manifold and let*

$f\colon \mathbb{R}^n \to N$ be a mapping of $\mathcal{K}$-bounded p-mean distortion having finite lower order λ. Then $\dim H^*(N) \le C$.

Although, the dependence of the constant C on the distortion and the finite lower of the mapping is not fully understood in general, it is explicitly known in some special cases. For $\lambda = 0$ the manifold N is a rational homology sphere, i.e. $\dim H^*(N) = 2$, by [14, Theorem 1]. In addition, for $n = 2, 3$, we have $\dim H^*(N) \le 2^n$ by Varopoulos's theorem and Poincaré duality. Let us also recall that by the rescaling principle for quasiregular mappings, see e.g. [2, Section 2], a quasiregularly elliptic manifold always admits quasiregular mappings having finite lower order at most n. Thus we recover the Bonk-Heinonen theorem from Theorem 1.

The proof of the main theorem relies on two ingredients of possible independent interest. In Section 2, we give a very simple proof for an extension of a special case of the Mattila-Rickman equidistribution theorem [13, Theorem 5.1]. For mappings of bounded mean distortion, our result reads as follows. In the statement of the theorem, the *logarithmic measure* $m_{\log}(E)$ of a set $E \subset (0, \infty)$ is

$$m_{\log}(E) = \int_E \frac{\mathrm{d}r}{r}.$$

Theorem 2. *Let N be a closed, connected, and oriented Riemannian n-manifold, $n \ge 2$, $u \in L^q(N)$, $q > n$, and suppose that $f\colon \mathbb{R}^n \to N$ is a mapping of bounded $(n-1)$-mean distortion. Then for every $\varepsilon > 0$ there exist $r_0 \ge 1$ and a set $F \subset [1, \infty)$ so that $m_{\log}([r/2, r] \setminus F) < \varepsilon$ for every $r \ge r_0$ and*

$$\lim_{\substack{r\to\infty \\ r\in F}} \frac{1}{A_f(r)} \int_{B^n(r)} (u \circ f)(x) J_f(x) \,\mathrm{d}x = \frac{1}{|N|} \int_N u(y) \,\mathrm{d}y, \tag{1.1}$$

where the integral on the right is with respect to the Riemannian measure of N and $|N|$ is the volume of N.

We find it interesting that the proof of this equidistribution theorem does not rely on discreteness and openness, as the sharper result of Mattila and Rickman for quasiregular mappings does, but uses only the change of variables methods. In fact, it is not known to us whether the mappings in question are discrete and open. A result of Manfredi and Villamor [11] states that mappings of finite distortion having distortion in L^p_{loc} for $p > n-1$ are discrete and open and hence branched covers. For $n = 2$, Iwaniec and Šverák [9] proved that distortion in L^p_{loc} for $p \ge n-1$ implies discreteness and openness. They also conjecture the same result in all dimensions $n > 2$. For recent results in this direction, see [6].

In Section 3 we consider Caccioppoli-type potential estimates for pull-backs of forms under mappings of finite distortion. Instead of focusing on the solutions of degenerate $\mathcal{A}$-harmonic equations arising in the pull-back, we consider pairs of closed forms (ξ, ζ) satisfying a nonnegativity condition $\star(\xi \wedge \zeta) \ge 0$, where $\star$ denotes the Hodge star duality operator. Such pairs arise naturally in linear and non-linear Hodge theory; pairs $(\xi, \star\xi)$ and $(\xi, \star|\xi|^{p-2}\xi)$ are nonnegative if ξ is a harmonic or a p-harmonic form, respectively. These pairs are special cases of *Cartan forms* as considered by Hajłasz, Iwaniec, Malý, and Onninen [4].

Having the equidistribution result and a Caccioppoli-type estimate at our disposal, we finish the proof of Theorem 1 in Section 4. The argument follows closely the proof of Bonk and Heinonen. The main difference is in the replacement of conformal exponents by exponents within a range determined by a Sobolev-embedding theorem for differential forms.

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2. A Mattila-Rickman type value distribution theorem

In this section we give a weak Mattila-Rickman type equidistribution theorem for mappings of finite distortion. Since the natural class of mappings in this theorem is larger than mappings of bounded mean distortion, we introduce first some notation.

Given an integrable function u on a Riemannian manifold N, we set

$$\int_N u = \int_N u \operatorname{vol}_N = \int_N u(y)\, \mathrm{d}\mu(y),$$

where vol_N is the volume form determined by the Riemannian metric of N and μ is the Riemannian measure on N. Submanifolds of $\mathbb{R}^n$ are assumed to inherit the standard Riemannian metric from $\mathbb{R}^n$. We also denote by

$$\fint_N u = \frac{1}{|N|}\int_N u$$

the mean value of function u over a Riemannian manifold N, where $|N|$ is the volume of N.

Let $f\colon \mathbb{R}^n \to N$ be a mapping of finite distortion. The outer distortion function of f gives rise to a type of logarithmic measure m_f on $(0,\infty)$ defined by

$$m_f(E) = \int_E \frac{\mathrm{d}r}{r k_f(r)},$$

where $k_f\colon (0,\infty) \to [1,\infty]$ is the *spherical mean distortion function*

$$k_f(r) = \left(\fint_{S^{n-1}(r)} K_f^{n-1} \right)^{\frac{1}{n-1}},$$

where $S^{n-1}(r)$ is the sphere of radius r about the origin in $\mathbb{R}^n$.

Remark 3. *For quasiregular mappings and mappings of bounded spherical mean distortion,* $1 \le k_f \le \mathcal{K}$*, the measure* m_f *is comparable, with a constant depending only on* $\mathcal{K}$*, to the* logarithmic measure $m_{\log}$,

$$m_{\log}(E) = \int_E \frac{\mathrm{d}r}{r}.$$

For a mapping of bounded mean distortion f*, the measures* m_f *and* $m_{\log}$ *need, however, not be comparable. A radial mapping* $f\colon \mathbb{R}^n \to \mathbb{R}^n$ *as in* [14, Example 14] *with* k_f *comparable to the function*

$$k(r) = 1 + \sum_{m=1}^{\infty} m\chi_{[2^m-1,2^m]}(r)$$

provides an example of a mapping of this type.

For a more detailed discussion on logarithmic measures in value distribution theory of quasiregular mappings, see e.g. [18, V.9.16].

The main theorem of this section reads as follows.

Theorem 4. *Let* N *be a closed, connected, and oriented Riemannian* n*-manifold and suppose that* $f\colon \mathbb{R}^n \to N$ *is a mapping of finite distortion such that* $m_f([1,\infty)) = \infty$*. Then for every* n*-form* ω *in* $L^q(\bigwedge^n N)$*,* $q > n$*, there exists a set* $E \subset [1,\infty)$ *of finite* m_f*-measure so that*

$$\frac{1}{A_f(r)} \int_{B^n(r)} f^*\omega \to \fint_N \omega \tag{2.2}$$

as $r \to \infty$*,* $r \notin E$*.*

Theorem 4 is analogous to the Euclidean version of the Mattila-Rickman equidistribution theorem [13, Theorem 5.11]. For our applications, it suffices to have the following version of this result.

Theorem 5. *Let* N *be a closed, connected, and oriented Riemannian* n*-manifold and suppose that* $f\colon \mathbb{R}^n \to N$ *is a mapping of finite distortion. Then for every* $\varepsilon > 0$ *and every* n*-form* ω *in* $L^q(\bigwedge^n N)$*,* $q > n$*, there exists a set* $E \subset [1,\infty)$ *of finite* m_f*-measure so that*

$$\left(\fint_N \omega - \varepsilon\right) \int_{B^n(r)} J_f < \int_{B^n(r)} f^*\omega < \left(\fint_N \omega + \varepsilon\right) \int_{B^n(r)} J_f \tag{2.3}$$

for $r \in [1,\infty) \setminus E$*.*

The proofs of Theorems 4 and 5 can be reduced to the following lemma corresponding to the case of exact n-forms.

Lemma 6. *Let* N *and* f *be as in Theorem 5. Then for every* $\delta > (n-1)/n$*,* $\varepsilon > 0$*, and every bounded* $(n-1)$*-form* τ *in* $W^{1,q}(\bigwedge^{n-1} N)$*,* $q > n$*, there exists a set* $E \subset [1,\infty)$ *of finite* m_f*-measure so that*

$$\left| \int_{S^{n-1}(r)} f^*\tau \right| < \varepsilon \left(\int_{B^n(r)} J_f \right)^{\delta} \tag{2.4}$$

for $r \in [1,\infty) \setminus E$.

The mappings we consider have the Lusin property (N) and hence support the change of variables formula, see e.g. [5], [12], and [4, Section 2.2]. We use these properties frequently in what follows.

Proof of Theorem 5 assuming Lemma 6. Let

$$\tilde{\omega} = \omega - \left(\fint_N \omega \right) \mathrm{vol}_N.$$

Since

$$\int_N \tilde{\omega} = 0,$$

$\tilde{\omega}$ is weakly exact; see e.g. [16, Section 3]. Thus, by the Poincaré inequality [8, Theorem 6.4], there exists an $(n-1)$-form $\tau \in W^{1,q}(\bigwedge^{n-1} N)$ so that $d\tau = \tilde{\omega}$. Since $q > n$, τ is Hölder continuous, and hence bounded, by the Sobolev embedding theorem.

Let $\varepsilon > 0$. Since

$$\int_{B^n(r)} f^*\tilde{\omega} = \int_{S^{n-1}(r)} f^*\tau$$

for almost every $r > 0$, we may apply Lemma 6 with $\delta = 1$ and we obtain a set $E \subset [1,\infty)$ of finite m_f-measure so that

$$\left| \int_{B^n(r)} f^*\tilde{\omega} \right| < \varepsilon \int_{B^n(r)} J_f$$

for $r \in [1,\infty) \setminus E$. Since

$$\int_{B^n(r)} f^*\tilde{\omega} = \int_{B^n(r)} f^*\omega - \left(\fint_N \omega \right) \int_{B^n(r)} J_f,$$

the claim follows. □

Proof of Theorem 4 assuming Lemma 6. Suppose first that the averaged counting function A_f is bounded. Let k be an integer so that the set $X = \{y \in N \colon \mathrm{card}\ f^{-1}(y) = k\}$ has positive measure. Thus

$$\int_{B^n(r)} (\chi_X \circ f) J_f = \int_X n(y, B^n(r); f) dy \to k|X|$$

as $r \to \infty$, where $|X|$ the Riemannian measure of X and $n(\cdot,\cdot; f)$ is the counting function,

$$n(y, B^n(r); f) = \mathrm{card}\ (f^{-1}(y) \cap B^n(r))$$

of f; see [12, Theorem 3.1].

We show next that $A_f(r) \to k|N|$ as $r \to \infty$. Let $A_\infty = \lim_{r\to\infty} A_f(r)$; the limit exists, since A_f is non-decreasing and bounded. Let $\varepsilon > 0$. By an application of Theorem 5 to $\chi_X \mathrm{vol}_N$, there exists a set $E \subset [1,\infty)$ of finite m_f-measure so that

$$\left(\frac{|X|}{|N|} - \varepsilon\right) \int_{B^n(r)} J_f < \int_{B^n(r)} (\chi_X \circ f)\, J_f < \left(\frac{|X|}{|N|} + \varepsilon\right) \int_{B^n(r)} J_f$$

for $r \in [1,\infty) \setminus E$. Since $m_f([1,\infty)) = \infty$, we obtain

$$\left(\frac{|X|}{|N|} - \varepsilon\right) A_\infty \le k|X| \le \left(\frac{|X|}{|N|} + \varepsilon\right) A_\infty.$$

Thus

$$\frac{k|N|}{1+\varepsilon|N|} \le A_\infty \le \frac{k|N|}{1-\varepsilon|N|}$$

and we have that $A_f(r) \to k|N|$ as $r \to \infty$. Hence $N \setminus X$ is a zero set and f is a k-to-1 map. Thus

$$\int_{\mathbb{R}^n} f^*\omega = k \int_N \omega = \left(\fint_N \omega \right) k|N|$$

for all $\omega \in L^q(N)$, $q > n$, by the change of variables.

Suppose now that the averaged counting function A_f is unbounded. Let ω and $\tilde\omega$ be n-forms as in the proof of Theorem 5. Then an application of Lemma 6 with $(n-1)/n < \delta < 1$ yields a set $E \subset [1,\infty)$ of finite m_f-measure so that

$$\left| \int_{B^n(r)} f^*\tilde\omega \right| < \left(\int_{B^n(r)} J_f \right)^{\delta}$$

for $r \in [1,\infty) \setminus E$. Thus

$$\left| \frac{1}{A_f(r)} \int_{B^n(r)} f^*\tilde\omega \right| < A_f(r)^{\delta-1} \to 0,$$

as $r \to \infty$, $r \notin E$. The claim follows. $\square$

Proof of Lemma 6. Let $E \subset [1,\infty)$ be the set of such radii $r \ge 1$ that (2.4) does not hold. Then, for almost every $r \in E$, Hölder's inequality yields

$$\begin{aligned}
\varepsilon \left(\int_{B^n(r)} J_f \right)^{\delta} &\le \int_{S^{n-1}(r)} |f^*\tau| \le \int_{S^{n-1}(r)} |Df|^{n-1} (|\tau| \circ f) \\
&\le \|\tau\|_\infty \left(\int_{S^{n-1}(r)} K_f^{n-1} \right)^{\frac{1}{n}} \left(\int_{S^{n-1}(r)} J_f \right)^{\frac{n-1}{n}}.
\end{aligned}$$

Thus the averaged counting function A_f satisfies the differential inequality

$$\varepsilon^{\frac{n}{n-1}} A_f(r)^{\delta\frac{n}{n-1}} \le C \|\tau\|_\infty^{\frac{n}{n-1}} r k_f(r) A_f'(r)$$

for almost every $r \in E$, where $C = C(n)$. Since

$$C \left(\frac{\|\tau\|_\infty}{\varepsilon} \right)^{\frac{n}{n-1}} \int_1^\infty \frac{A_f'(r)}{A_f(r)^{\delta\frac{n}{n-1}}}\, \mathrm{d}r \quad \ge \quad \int_E \frac{\mathrm{d}r}{r k_f(r)} = m_f(E)$$

by the differential inequality and

$$\int_1^\infty \frac{A_f'(r)}{A_f(r)^{\delta\frac{n}{n-1}}}\,\mathrm{d}r \quad \le \quad \frac{1}{\delta\frac{n}{n-1}-1} A_f(1)^{1-\delta\frac{n}{n-1}},$$

the claim follows. $\square$

To prove Theorem 2 we consider first the following lemma.

Lemma 7. *Let $f\colon \mathbb{R}^n \to N$ be a mapping of $\mathcal{K}$-bounded $(n-1)$-mean distortion. Given $\varepsilon > 0$ there exists $\delta > 0$, $R_0 \ge 1$, and a set $F \subset [1,\infty)$ so that*

$$m_{\log}([R/2,R] \setminus F) < \varepsilon$$

for $R \ge R_0$ and

$$m_f(E) \ge \delta m_{\log}(E)$$

for every measurable set $E \subset F$. In particular, $m_f([1,\infty)) = \infty$.

Proof. Let $\delta > 0$ and consider the set

$$B_\delta = \{r \in (0,\infty)\colon k_f(r) > 1/\delta\}.$$

Then

$$m_f(E) = \int_E \frac{\mathrm{d}r}{r k_f(r)} \ge \delta \int_E \frac{\mathrm{d}r}{r} = \delta m_{\log}(E)$$

for every measurable set $E \subset (0,\infty) \setminus B_\delta$.

Fix $r_0 \ge 1$ so that

$$\int_{B^n(r)} K_f^{n-1} \le \mathcal{K}^{n-1} |B^n(r)|$$

for $r \ge r_0$ and set $R_0 = 2r_0$. It suffices to show that

$$|B_\delta \cap [R/2,R]| < C\delta^{n-1} R$$

for $R \ge R_0$, where $C = C(n,\mathcal{K})$.

Let $R \ge R_0$ and denote $B_{\delta,R} = B_\delta \cap [R/2,R]$. Since $k_f(r)\delta > 1$ for $r \in B_\delta$, we have

$$\begin{aligned}(R/2)^{n-1}|B_{\delta,R}| \quad &\le \quad \int_{B_{\delta,R}} r^{n-1}\,\mathrm{d}r \le \delta^{n-1} \int_{B_{\delta,R}} r^{n-1} k_f(r)^{n-1}\,\mathrm{d}r \\ &\le \quad \delta^{n-1} \int_{B^n(R)} K^{n-1} \le \delta^{n-1}\mathcal{K}^{n-1}|B^n(R)|.\end{aligned}$$

Thus

$$|B_{\delta,R}| \le C\delta^{n-1} R,$$

where $C = C(n,\mathcal{K})$. $\square$

Proof of Theorem 2. Let $\omega = u \,\mathrm{vol}_N$ and let $E \subset [1,\infty)$ be a set of finite m_f-measure as in Theorem 5. Given $\varepsilon > 0$, we may fix, by Lemma 7, $\delta > 0$, $R_1 \ge 1$, and a set $F' \subset [1,\infty)$ so that

$$m_{\log}([R/2,R] \setminus F') < \varepsilon/2$$

for $R \geq R_1$, and

$$m_f(E \cap F') \geq \delta m_{\log}(E \cap F').$$

We fix $R_0 \geq R_1$ so that $m_{\log}(E \cap [R_0/2, \infty)) < \varepsilon/2$. Then R_0 and $F = F' \setminus E$ satisfy conditions of the claim. $\square$

3. A Caccioppoli-type estimate

In this section we deduce a counterpart for the Caccioppoli-type estimate used in the proof of Bonk and Heinonen. In what follows we use the notation q' to denote the Hölder conjugate $q/(q-1)$ of $q > 1$.

We begin with a potential theoretic lemma of Caccioppoli-type for non-negative pairs of forms.

Lemma 8. *Let $0 < r < R$, $1 \leq \ell \leq n-1$, and $q > 1$. Let ω and ω' be closed forms in $L^q(\bigwedge^\ell B^n(R))$ and $L^{q'}(\bigwedge^{n-\ell} B^n(R))$, respectively, so that $\star(\omega \wedge \omega') \geq 0$. Then there exists $C = C(n) > 0$ so that*

$$\int_{B^n(r)} \omega \wedge \omega' \leq \frac{C}{R-r} \left(\int_{B^n(R)} |\tau|^q \right)^{1/q} \left(\int_{B^n(R)} |\omega'|^{q'} \right)^{1/q'} \tag{3.5}$$

for every $\tau \in W^{d,q}(\bigwedge^{\ell-1} B^n(R))$ satisfying $d\tau = \omega$.

Proof. Since $\star(\omega \wedge \omega') \geq 0$, we have, by Stokes' theorem and Hadamard's and Hölder's inequalities,

$$\begin{aligned}
\int_{B^n(r)} \omega \wedge \omega' &\leq \frac{1}{R-r} \int_r^R \int_{B^n(t)} \omega \wedge \omega' = \frac{1}{R-r} \int_r^R \int_{S^{n-1}(t)} \tau \wedge \omega' \\
&\leq \frac{1}{R-r} \int_r^R \int_{S^{n-1}(t)} |\tau \wedge \omega'| \leq \frac{C}{R-r} \int_{B^n(R)} |\tau||\omega'| \\
&\leq \frac{C}{R-r} \left(\int_{B^n(R)} |\tau|^q \right)^{1/q} \left(\int_{B^n(R)} |\omega'|^{q'} \right)^{1/q'},
\end{aligned}$$

where $C = C(n) > 0$. $\square$

The main result of this section combines the Caccioppoli-type estimate with value distribution results for mappings of finite distortion having finite lower order. The proof uses the following observation, typical in value distribution theory; see e.g. [2, Lemma 4.14]. For the reader's convenience we give a simple proof.

Lemma 9. *Let $\lambda > 0$ and $\varphi\colon (0,\infty) \to (0,\infty)$ be a nondecreasing function so that*

$$\lambda = \liminf_{r \to \infty} \frac{\log \varphi(r)}{\log r} < \infty. \tag{3.6}$$

Then for every $r_0 > 0$ there exists $r_1 \geq r_0$ so that

$$\varphi(r) \leq 5^\lambda \varphi(r/2) \tag{3.7}$$

for all $r_1/2 \le r \le r_1$. In particular, (3.7) *holds in a set of infinite logarithmic measure.*

Proof. Let $r_0 > 0$. We show first that there exists $r_1 \ge r_0$ so that $\varphi(r_1) \le 5^\lambda \varphi(r_1/4)$. Should this not be the case, $\varphi(4^k r_0) \ge 5^{k\lambda}\varphi(r_0)$ for every $k \ge 0$. To show that this is a contradiction, let $k_0 > 0$ to be fixed later. Let $k \ge k_0$ and $4^k r_0 \le r \le 4^{k+1} r_0$. Then

$$\frac{\log \varphi(r)}{\log(r)} \ge \frac{\log(\varphi(4^k r_0))}{\log(4^{k+1} r_0)} \ge \frac{\log(5^{k\lambda}\varphi(r_0))}{\log(4^k 4 r_0)} = \frac{k\lambda \log 5 + \log \varphi(r_0)}{k \log 4 + \log(4r_0)} > C\lambda,$$

where $C = C(k_0) > 1$ for k_0 large. This contradicts (3.6). Thus there exists $r_1 \ge r_0$ so that $\varphi(r_1) \le 5^\lambda \varphi(r_1/4)$. Then

$$\varphi(r) \le \varphi(r_1) \le 5^\lambda \varphi(r_1/4) \le 5^\lambda \varphi(r/2).$$

for every $r_1/2 \le r \le r_1$. $\square$

The following proposition combines the Caccioppoli-type estimate with value distribution and finite lower order.

Proposition 10. *Let N be a closed, connected, and oriented Riemannian n-manifold so that $|N| = 1$, and let $f\colon \mathbb{R}^n \to N$ be a mapping of $\mathcal{K}$-bounded p-mean distortion for $p > n-1$ having finite lower order λ. Then there exists a sequence (r_i) tending to infinity so that the following holds.*

Let $1 \le \ell \le n-1$ and suppose that $q > n/\ell$ satisfies

$$q' = \frac{p}{p+1}\left(\frac{n}{\ell}\right)'.$$

Let also ξ and ζ be closed forms in $L^\infty(\bigwedge^\ell N)$ and in $L^\infty(\bigwedge^{n-\ell} N)$, respectively, so that $\star(\xi \wedge \zeta) \ge 0$. There exist a constant $C = C(n, \ell, \lambda, \|\xi \wedge \zeta\|_1, \|\zeta\|_{\frac{n}{n-\ell}}) > 0$ and a set $E \subset [1,\infty)$ of finite m_f-measure so that for every $\alpha \in W^{d,q}_{\mathrm{loc}}\left(\bigwedge^{\ell-1} B^n(r)\right)$ satisfying $d\alpha = f^*\xi$ we have

$$\left(\int_{B^n(r)} J_f\right)^{\ell/n} \le C\mathcal{K}^{\frac{n-\ell}{n}} r^{\ell-1-\frac{n}{q}} \left(\int_{B^n(r)} |\alpha|^q\right)^{1/q}$$

for $r \in (\bigcup_i [r_i/2, r_i]) \setminus E$.

Proof. Since f has finite lower order λ, there exists, by Lemma 9, $C = C(\lambda) > 0$ and a sequence (r_i) tending to infinity so that

$$A_f(r) \le C A_f(r/2) \tag{3.8}$$

for $r_i/2 \le r \le r_i$ and every i. We show that this sequence satisfies conditions of the claim.

We show first that

$$\tag{3.9}\begin{aligned}&\int_{B^n(r)} f^*\xi\wedge f^*\zeta \\ &\quad\le \frac{C}{R-r}\left(\int_{B^n(R)} K_f^{\frac{1}{s-1}}\right)^{\frac{s-1}{sq'}}\left(\int_{B^n(R)}|\alpha|^q\right)^{1/q}\left(\int_{B^n(R)}(|\zeta|\circ f)^{\frac{n}{n-\ell}}J_f\right)^{\frac{n-\ell}{n}}\end{aligned}$$

for $R>r\ge 1$, where $s=\frac{n}{n-\ell}\frac{1}{q'}$.

Since $\star(f^*\xi\wedge f^*\zeta)=\star(\xi\wedge\zeta)\circ f J_f\ge 0$, it suffices, by Lemma 8, to show that

$$\left(\int_{B^n(R)}|f^*\zeta|^{q'}\right)^{1/q'}\le\left(\int_{B^n(R)} K_f^{\frac{1}{s-1}}\right)^{\frac{s-1}{sq'}}\left(\int_{B^n(R)}(|\zeta|\circ f)^{\frac{n}{n-\ell}}J_f\right)^{\frac{n-\ell}{n}}.$$

By Hölder's inequality, we obtain

$$\begin{aligned}\int_{B^n(R)}|f^*\zeta|^{q'} &\le \int_{B^n(R)}(K_fJ_f)^{\frac{n-\ell}{n}q'}(|\zeta|\circ f)^{q'}\\ &= \int_{B^n(R)} K_f^{\frac{1}{s}}(|\zeta|\circ f)^{q'}J_f^{\frac{1}{s}}\\ &\le \left(\int_{B^n(R)} K_f^{\frac{1}{s-1}}\right)^{\frac{s-1}{s}}\left(\int_{B^n(R)}(|\zeta|\circ f)^{\frac{n}{n-\ell}}J_f\right)^{\frac{1}{s}}.\end{aligned}$$

The inequality (3.9) now follows.

By Theorem 5, we fix a set $E\subset[1,\infty)$ of finite m_f-measure so that

$$\tag{3.10}\frac{1}{2C}\int_{B^n(r/2)}J_f\le\int_{B^n(r/2)} f^*\xi\wedge f^*\zeta$$

and

$$\tag{3.11}\int_{B^n(r)}(|\zeta|\circ f)^{\frac{n}{n-\ell}}J_f\le 2C\int_{B^n(r)}J_f$$

for $r\in[1,\infty)\setminus E$, where $C=C(n,\ell,\|\xi\wedge\zeta\|_1,\|\zeta\|_{\frac{n}{n-\ell}})>0$.

Thus (3.9) and (3.8) together with (3.10) yield

$$\begin{aligned}&\int_{B^n(r)}J_f\le C\int_{B^n(r/2)}J_f\le C\int_{B^n(r/2)}f^*\xi\wedge f^*\zeta\\ &\quad\le\frac{C}{r}\left(\int_{B^n(r)}K_f^{\frac{1}{s-1}}\right)^{\frac{s-1}{sq'}}\left(\int_{B^n(r)}|\alpha|^q\right)^{1/q}\left(\int_{B^n(r)}(|\zeta|\circ f)^{\frac{n}{n-\ell}}J_f\right)^{\frac{n-\ell}{n}}\\ &\quad\le\frac{C}{r}\left(\int_{B^n(r)}K_f^{\frac{1}{s-1}}\right)^{\frac{s-1}{sq'}}\left(\int_{B^n(r)}|\alpha|^q\right)^{1/q}\left(\int_{B^n(r)}J_f\right)^{\frac{n-\ell}{n}},\end{aligned}$$

for $r\in(\bigcup_i[r_i/2,r_i])\setminus E$, where $C=C(n,\ell,\lambda,\|\xi\wedge\zeta\|_1,\|\zeta\|_{\frac{n}{n-\ell}})>0$.

Since q satisfies $q' = \frac{p}{p+1}\left(\frac{n}{\ell}\right)'$, we have

$$\frac{1}{s-1} = \frac{1}{\frac{n}{n-\ell}\frac{1}{q'} - 1} = \frac{q'}{\left(\frac{n}{\ell}\right)' - q'} = p$$

and

$$\left(\int_{B^n(r)} K_f^{\frac{1}{s-1}}\right)^{\frac{s-1}{sq'}} = \left(\int_{B^n(r)} K_f^p\right)^{\frac{1}{sq'p}} \le C\mathcal{K}^{\frac{n-\ell}{n}} r^{\frac{n}{sq'p}} = C\mathcal{K}^{\frac{n-\ell}{n}} r^{\frac{n-\ell}{p}},$$

for every r large enough, where $C = C(n,\ell,p)$. This concludes the proof. $\square$

4. Proof of Theorem 1

By Poincaré duality, it suffices to consider the cohomology groups $H^\ell(N)$ for $1 \le \ell \le n/2$. We may also assume that $|N| = 1$.

Suppose $d = \dim H^\ell(N) > 0$. By non-linear Hodge theory [20], we may fix (n/ℓ)-harmonic ℓ-forms $\xi_1, \ldots, \xi_d$ on N so that the cohomology classes of the forms span $H^\ell(N)$ and that the forms satisfy

$$\|\xi_i\|_{n/\ell} = 1 \quad \text{and} \quad \|\xi_i - \xi_j\|_{n/\ell} \ge 1$$

for all i and $j \ne i$. For every i, we set ζ_i to be the (n/ℓ)-harmonic conjugate of ξ_i, i.e., $\zeta_i = \star|\xi_i|^{\frac{n}{\ell}-2}\xi_i$. Then $\|\zeta_i\|_{\frac{n}{n-\ell}} = 1$. The forms ξ_i and ζ_i are Hölder continuous by results of Uhlenbeck [21] and Ural'tseva [22]. Especially, they are bounded.

Since $n/\ell \ge 2$, we have, by a pointwise monotonicity estimate (see e.g. [1, p. 288]), that

$$\int_N (\xi_i - \xi_j) \wedge (\zeta_i - \zeta_j) \ge C \int_N |\xi_i - \xi_j|^{n/\ell} \ge C,$$

where $C = C(n) > 0$. By Hölder's inequality, we also obtain

$$\int_N (\xi_i - \xi_j) \wedge (\zeta_i - \zeta_j) \le C\|\xi_i - \xi_j\|_{n/\ell}\left(\|\zeta_i\|_{\frac{n}{n-\ell}} + \|\zeta_j\|_{\frac{n}{n-\ell}}\right) \le C,$$

where $C = C(n) > 0$. For brevity, we set $\xi_{ij} = \xi_i - \xi_j$ and $\zeta_{ij} = \zeta_i - \zeta_j$ for every $i \ne j$.

To obtain an estimate for the number of forms ξ_i, we use the compactness of the Poincaré homotopy operator $T\colon L^s(\bigwedge^\ell B^n) \to L^q(\bigwedge^{\ell-1} B^n)$ of Iwaniec and Lutoborski [7]. Since T is a composition of a continuous operator $L^s(\bigwedge^\ell B^n) \to W^{1,s}(\bigwedge^{\ell-1} B^n)$ and a Sobolev embedding $W^{1,s}(\bigwedge^{\ell-1} B^n) \to L^q(\bigwedge^{\ell-1} B^n)$, we have that T is compact for $s \le q < s^*$, where $s^* = ns/(n-s)$ is the Sobolev conjugate of s. Thus it suffices to show that there exist exponents s and q and a radius $r > 0$ so that $s \le q < s^*$ and that we have the estimates

$$\|T\lambda_r^* f^* \xi_i\|_q \le C\left(\int_{B^n(r)} J_f\right)^{\ell/n}$$

and

$$\|T\lambda_r^* f^*\xi_i - T\lambda_r^* f^*\xi_j\|_q \geq \frac{1}{C}\left(\int_{B^n(r)} J_f\right)^{\ell/n},$$

where $C = C(n,\lambda,\mathcal{K}) > 0$ and $\lambda_r \colon \mathbb{R}^n \to \mathbb{R}^n$ is the similarity mapping $x \mapsto rx$. Then, by the compactness of T, the number of forms $\lambda_r^* f^*\xi_i$, and hence also ξ_i, is bounded by a constant depending only on n, λ, and $\mathcal{K}$.

We fix

$$s = \frac{1}{2}\left(\frac{n}{\ell+1} + \frac{n}{\ell}\right) \quad \text{and} \quad q = \frac{1}{2}\left(\frac{n}{\ell} + s^*\right).$$

Since $s > n/(\ell+1)$, we have $s^* > n/\ell$ and $n/\ell < q < s^*$. Thus $q' < (n/\ell)'$ and there exists $\tilde{p} > 1$ so that

$$q' = \frac{\tilde{p}}{\tilde{p}+1}\left(\frac{n}{\ell}\right)'.$$

Since $1 \leq \ell \leq n/2$, we may fix $p = p(n) > n-1$ so that

$$p \geq \max\left\{\tilde{p}, \frac{s\ell}{n - s\ell}\right\}.$$

Suppose now that $f \colon \mathbb{R}^n \to N$ is a mapping of $\mathcal{K}$-bounded p-mean distortion. By Theorem 5 and Lemma 7, we fix $R_0 \geq 1$, $\delta > 0$, and a set $F \subset [1,\infty)$ so that $m_{\log}(F \cap [R/2,R]) > (5/6)\log 2$ for $R \geq R_0$, $m_f(E) \geq \delta m_{\log}(E)$ for a measurable set $E \subset F$, and

$$\int_{B^n(r)} (|\xi_i| \circ f)^{n/\ell} J_f \leq C \int_{B^n(r)} J_f$$

for every i and every $r \in F$, where $C = C(n) > 0$.

Using Hölder's inequality, we obtain

(4.12)

$$\begin{aligned}\left(\int_{B^n(r)} |f^*\xi_i|^s\right)^{1/s} &\leq \left(\int_{B^n(r)} K_f^{\frac{\ell s}{n-\ell s}}\right)^{\frac{n-\ell s}{ns}} \left(\int_{B^n(r)} (|\xi_i| \circ f)^{n/\ell} J_f\right)^{\ell/n} \\ &\leq C\left(\fint_{B^n(r)} K_f^p\right)^{\frac{\ell}{np}} r^{\frac{n-\ell s}{s}} \left(\int_{B^n(r)} J_f\right)^{\ell/n} \\ &\leq C\mathcal{K}^{\frac{\ell}{n}} r^{\frac{n-\ell s}{s}} \left(\int_{B^n(r)} J_f\right)^{\ell/n}\end{aligned}$$

for every $r \in F$, where $C = C(n) > 0$. Thus

$$\left(\int_{B^n} |\lambda_r^* f^*\xi_i|^s\right)^{1/s} \leq C\mathcal{K}^{\frac{\ell}{n}} \left(\int_{B^n(r)} J_f\right)^{\ell/n}$$

for every $r \in F$.

Since

$$dT\lambda_r^* f^*\xi_{ij} = \lambda_r^* f^*\xi_{ij} - Td\lambda_r^* f^*\xi_{ij} = \lambda_r^* f^*\xi_{ij},$$

we may set $\alpha_{ij} = \lambda_{1/r}^* T \lambda_r^* f^* \xi_{ij}$ and we have

$$d\alpha_{ij} = f^*\xi_{ij}.$$

for every $i \neq j$. By Proposition 10, there exists a sequence (r_i) tending to infinity, a set $E \subset [1,\infty)$ of finite m_f-measure, and $C = C(n,\lambda) > 0$ so that

$$\left(\int_{B^n(r)} J_f \right)^{\ell/n} \le C \mathcal{K}^{\frac{n-\ell}{n}} r^{\ell-1-\frac{n}{q}} \left(\int_{B^n(r)} |\alpha_{ij}|^q \right)^{1/q} \tag{4.13}$$

for $r \in (\bigcup_i [r_i/2, r_i]) \setminus E$ and all $i \neq j$.

Since E has finite m_f-measure, $m_{\log}(F \cap E) < \infty$ and we may fix $r \in F \cap (\bigcup_i [r_i/2, r_i]) \setminus E$ so that (4.12) and (4.13) hold for every $i \neq j$.

Since

$$\left(\int_{B^n} |T\lambda_r^* f^* \xi_{ij}|^q \right)^{1/q} = r^{\ell-1-\frac{n}{q}} \left(\int_{B^n(r)} |\alpha_{ij}|^q \right)^{1/q},$$

we obtain the last required estimate

$$\|T\lambda_r^* f^* \xi_{ij}\|_q \ge C \left(\int_{B^n(r)} J_f \right)^{\ell/n}.$$

This concludes the proof of Theorem 1.

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