Advanced Review

Fuzzy set theory

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Since its inception in 1965, the theory of fuzzy sets has advanced in a variety of ways and in many disciplines. Applications of this theory can be found, for example, in artificial intelligence, computer science, medicine, control engineering, decision theory, expert systems, logic, management science, operations research, pattern recognition, and robotics. Mathematical developments have advanced to a very high standard and are still forthcoming to day. In this review, the basic mathematical framework of fuzzy set theory will be described, as well as the most important applications of this theory to other theories and techniques. Since 1992 fuzzy set theory, the theory of neural nets and the area of evolutionary programming have become known under the name of 'computational intelligence' or 'soft computing'. The relationship between these areas has naturally become particularly close. In this review, however, we will focus primarily on fuzzy set theory. Applications of fuzzy set theory to real problems are abound. Some references will be given. To describe even a part of them would certainly exceed the scope of this review. *WIREs Comp Stat* 2010 2 317–332

Most of our traditional tools for formal modeling, reasoning, and computing are crisp, deterministic, and precise in character. Crisp means dichotomous, that is, yes-or-no type rather than more-or-less type. In traditional dual logic, for instance, a statement can be true or false—and nothing in between. In set theory, an element can either belong to a set or not; in optimization a solution can be feasible or not. Precision assumes that the parameters of a model represent exactly the real system that has been modeled. This, generally, also implies that the model is unequivocal, that is, that it contains no ambiguities. Certainty eventually indicates that we assume the structures and parameters of the model to be definitely known and that there are no doubts about their values or their occurrence. Unluckily these assumptions and beliefs are not justified if it is important, that the model describes well reality (which is neither crisp nor certain). In addition, the complete description of a real system would often require far more detailed data than a human being could ever recognize simultaneously, process, and understand. This situation has already been recognized by thinkers in the past. In 1923, the philosopher B. Russell referred to the first point when he wrote: 'All traditional logic habitually assumes that precise symbols are being employed. It is, therefore, not applicable to this terrestrial life but only to an imagined celestial existence'.[1] L. Zadeh referred to the second point, when he wrote: 'As the complexity of a system increases, our ability to make precise and yet significant statements about its behavior diminishes until a threshold is reached beyond which precision and significance (or relevance) become almost mutually exclusive characteristics'.[2] For a long time, probability theory and statistics have been the predominant theories and tools to model uncertainties of reality.[3,4] They are based—as all formal theories—on certain axiomatic assumptions, which are hardly ever tested, when these theories are applied to reality. In the meantime more than 20 other 'uncertainty theories' have been developed,[5] which partly contradict each other and partly complement each other. Fuzzy set theory—formally speaking—is one of these theories, which was initially intended to be an extension of dual logic and/or classical set theory. During the last decades, it has been developed in the direction of a powerful 'fuzzy' mathematics. When it is used, however, as a tool to model reality better than traditional theories, then an empirical validation is very desirable.[6] In the following sections, the formal theory is described and some of the attempts to verify the theory empirically in reality will be summarized.

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HISTORY

A short historical review may be useful to better understand the character and motivation of this theory. The first publications in fuzzy set theory by Zadeh[7] and Goguen[8] show the intention of the authors to generalize the classical notion of a set and a proposition to accommodate fuzziness in the sense that it is contained in human language, that is, in human judgment, evaluation, and decisions. Zadeh writes: 'The notion of a fuzzy set provides a convenient point of departure for the construction of a conceptual framework which parallels in many respects the framework used in the case of ordinary sets, but is more general than the latter and, potentially, may prove to have a much wider scope of applicability, particularly in the fields of pattern classification and information processing. Essentially, such a framework provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables'.[7] 'Imprecision' here is meant in the sense of vagueness rather than the lack of knowledge about the value of a parameter (as in tolerance analysis). Fuzzy set theory provides a strict mathematical framework (there is nothing fuzzy about fuzzy set theory!) in which vague conceptual phenomena can be precisely and rigorously studied. It can also be considered as a modeling language, well suited for situations in which fuzzy relations, criteria, and phenomena exist. The acceptance of this theory grew slowly in the 1960s and 1970s of the last century. In the second half of the 1970s, however, the first successful practical applications in the control of technological processes via fuzzy rule-based systems, called fuzzy control (heating systems, cement factories, etc.), boosted the interest in this area considerably. Successful applications, particularly in Japan, in washing machines, video cameras, cranes, subway trains, and so on triggered further interest and research in the 1980s so that in 1984 already approximately 4000 publications existed and in 2000 more than 30,000. Roughly speaking, fuzzy set theory during the last decades has developed along two lines:

1. As a formal theory that, when maturing, became more sophisticated and specified and was enlarged by original ideas and concepts as well as by 'embracing' classical mathematical areas, such as algebra,[9,10] graph theory,[11] topology, and so on by generalizing or 'fuzzifying' them. This development is still ongoing.
2. As an application-oriented 'fuzzy technology', that is, as a tool for modeling, problem solving, and data mining that has been proven superior to existing methods in many cases and as attractive 'add-on' to classical approaches in other cases.

In 1992, in three simultaneous conferences in Europe, Japan, and the United States, the three areas of fuzzy set theory, neural nets, and evolutionary computing (genetic algorithms) joined forces and are henceforth known under 'computational intelligence'.[8,12,13] In a similar way, the term 'soft computing' is used for a number of approaches that deal essentially with uncertainty and imprecision.

MATHEMATICAL THEORY AND EMPIRICAL EVIDENCE

Basic Definitions and Operations

The axiomatic bases of fuzzy set theory are manifold. Gottwald offers a good review.[14–16] We shall concentrate on the elements of the theory itself:

Definition 1 *If X is a collection of objects denoted generically by x, then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs:*

$$\tilde{A} = \left\{ (x, \mu_{\tilde{A}}(x)|x \in X) \right\} \tag{1}$$

$\mu_{\tilde{A}}(x)$ is called the membership function (generalized characteristic function) which maps X to the membership space M. Its range is the subset of nonnegative real numbers whose supremum is finite. For $\sup \mu_{\tilde{A}}(x) = 1$: normalized fuzzy set.

In Definition 1, the membership function of the fuzzy set is a crisp (real-valued) function. Zadeh also defined fuzzy sets in which the membership functions themselves are fuzzy sets. Those sets can be defined as follows:

Definition 2 *A type m fuzzy set is a fuzzy set whose membership values are type $m-1$, $m > 1$, fuzzy sets on* $[0, 1]$.

Because the termination of the fuzzification on stage $r \leq m$ seems arbitrary or difficult to justify, Hirota[17] defined a fuzzy set the membership function of which is pointwise a probability distribution: the probabilistic set.

Definition 3 *A probabilistic set*[17] *A on X is defined by defining function* μ_A,

$$\mu_A : X \times \Omega \ni (x, \omega) \to \mu_A(x, \omega) \in \Omega_C \tag{2}$$

and $(\Omega_C, B_C) = [0, 1]$ *are Borel sets.*

Definition 4 *A linguistic variable*[2] *is characterized by a quintuple* $(x, T(x), U, G, \tilde{M})$, *in which x is the name of the variable, T(x) (or simply T) denotes the term set of x, that is, the set of names of linguistic values of x. Each of these values is a fuzzy variable, denoted generically by X and ranging over a universe of discourse U, which is associated with the base variable u; G is a syntactic rule (which usually has the form of a grammar) for generating the name, X, of values of x. M is a semantic rule for associating with each X its meaning.* $\tilde{M}(X)$ *is a fuzzy subset of U. A particular X, that is, a name generated by G, is called a term (e.g., Figure 1).*

Of course, the fuzzy sets in Figure 1 representing the values of the linguistic variable 'AGE' may more appropriately be continuous fuzzy sets.

Definition 5 *A fuzzy number* $\tilde{M}$ *is a convex normalized fuzzy set* $\tilde{M}$ *of the real line R such that*

1. *it exists exactly one* $x_0 \in \mathbb{R}$ $\mu_{\tilde{M}}(x_0) = 1$ *(*x_0 *is called the mean value of* $\tilde{M}$*).*
2. $\mu_{\tilde{M}}(x)$ *is piecewise continuous.*

Dubois and Prade[9] defined LR-fuzzy numbers as follows:

Definition 6 *A fuzzy number* $\tilde{M}$ *is of LR-type if there exist reference functions L (for left) and R (for right), and scalars* $\alpha > 0$, $\beta > 0$, *with*

$$\mu_{\tilde{M}}(x) = L\left(\frac{m-x}{\alpha}\right) \text{ for } x \le m \tag{3}$$

$$= R\left(\frac{x-m}{\beta}\right) \text{ for } x \le m \tag{4}$$

(see Figure 2)

Some other useful definitions are the following:

Definition 7 *The support of a fuzzy set* $\tilde{A}$, $S(\tilde{A})$ *is the crisp set of all* $x \in X$ *such that* $\mu_{\tilde{A}}(x) > 0$.

The (crisp) set of elements that belong to the fuzzy set $\tilde{A}$ *at least to the degree* α *is called the* α*-level set:*

$$A_\alpha = \left\{x \in X | \mu_{\tilde{A}} \ge \alpha\right\} \tag{5}$$

$A'_\alpha = \left\{x \in X | \mu_{\tilde{A}} > \alpha\right\}$ *is called strong* α*-level set or strong* α*-cut.*

Definition 8 *A fuzzy set* $\tilde{A}$ *is convex if*

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \ge \min\left\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\right\},$$
$$x_1, x_2 \in X, \lambda \in [0, 1]. \tag{6}$$

Alternatively, a fuzzy set is convex if all α*-level sets are convex.*

Definition 9 *For a finite fuzzy set* $\tilde{A}$, *the cardinality* $|\tilde{A}|$ *is defined as*

$$|\tilde{A}| = \sum_{x \in X} \mu_{\tilde{A}}(x). \tag{7}$$

$||\tilde{A}|| = \frac{|\tilde{A}|}{|X|}$ *is called the relative cardinality of* $\tilde{A}$.

A variety of definitions exist for fuzzy measures and measures of fuzziness. The interested reader is referred to Refs 18–21.

Operations on Fuzzy Sets

In his first publication, Zadeh[7] defined the following operations for fuzzy sets as generalization of crisp sets and of crisp statements (the reader should realize that the set theoretic operations intersection, union and complement correspond to the logical operators and, inclusive or and negation):

Definition 10 *Intersection (logical and): the membership function of the intersection of two fuzzy sets* $\tilde{A}$ *and* $\tilde{B}$ *is defined as:*

$$\mu_{\tilde{A} \cap \tilde{B}}(X) = \mathrm{Min}(\mu_{\tilde{A}}(X), \mu_{\tilde{B}}(X)) \forall x \in X \tag{8}$$

Definition 11 *Union (exclusive or): the membership function of the union is defined as:*

$$\mu_{\tilde{A} \cup \tilde{B}}(X) = \mathrm{Max}(\mu_{\tilde{A}}(X), \mu_{\tilde{B}}(X)) \forall x \in X \tag{9}$$

Definition 12 *Complement (negation): the membership function of the complement is defined as:*

$$\mu_{\tilde{A}}(X) = 1 - \mu_{\tilde{A}}(X) \forall x \in X \tag{10}$$

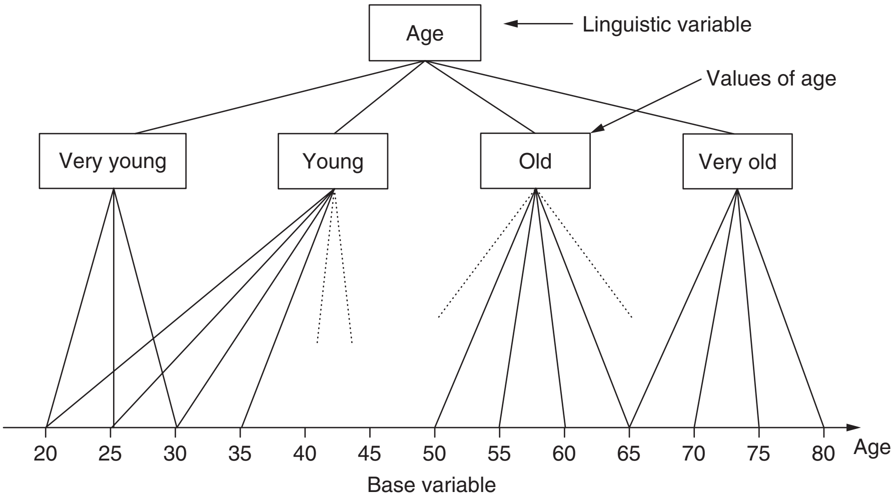

FIGURE 1 | Linguistic variable 'Age'.[2]

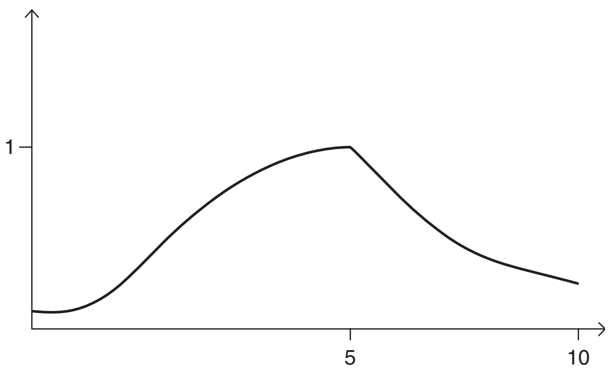

FIGURE 2 | LR-representation of a fuzzy number.

These definitions were later extended. The 'logical and' (intersection) can also be modeled as a t-norm[18,21–28] and the 'inclusive or' (union) as a t-conorm. Both types are monotonic, commutative, and associative.

Typical dual pairs of nonparameterized t-norms and t-conorms are compiled below:

(**1a**) drastic product:

$$t_w\left(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right) = \begin{cases} \min\left\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right\} & \text{if } \max\left\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right\} = 1 \\ 0 & \text{otherwise} \end{cases} \quad (11)$$

(**1b**) drastic sum:

$$s_w\left(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right) = \begin{cases} \max\left\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right\} & \text{if } \min\left\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right\} = 0 \\ 1 & \text{otherwise} \end{cases} \quad (12)$$

(**2a**) bounded difference:

$$t_1(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \max\left\{0, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - 1\right\} \quad (13)$$

(**2b**) bounded sum:

$$s_1(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \min\left\{1, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x)\right\} \quad (14)$$

(**3a**) Einstein product:

$$t_{1.5}(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \frac{\mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)}{2 - \left[\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)\right]} \quad (15)$$

(**3b**) Einstein sum:

$$s_{1.5}(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \frac{\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x)}{1 + \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)} \quad (16)$$

(**4a**) Hamacher product:

$$t_{2.5}(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \frac{\mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)}{\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)} \quad (17)$$

(**4b**) Hamacher sum:

$$s_{2.5}(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \frac{\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - 2\mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)}{1 - \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)} \quad (18)$$

(5a) minimum:

$$t_3(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \min\left\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right\} \quad (19)$$

(5b) maximum:

$$s_3(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \max\left\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right\}. \quad (20)$$

These operators are ordered as follows:

$$t_w \le t_1 \le t_{1.5} \le t_2 \le t_{2.5} \le t_3 \quad (21)$$
$$s_3 \le s_{2.5} \le s_2 \le s_{1.5} \le s_1 \le s_w. \quad (22)$$

Also a number of parameterized operators were defined, such as:

1. Hamacher union:

$$\mu_{\tilde{A}\cup\tilde{B}}(x) = \frac{(\gamma' - 1)\mu_{\tilde{B}}(x) + \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x)}{1 + \gamma'\mu_{\tilde{A}}(x)\mu_{\tilde{B}}(x)} \quad (23)$$

2. Yager intersection:

$$\mu_{\tilde{A}\cap\tilde{B}}(x) = 1 - \min\left\{1, \left((1-\mu_{\tilde{A}(x)})^p + (1-\mu_{\tilde{B}(x)})^p\right)^{1/p}\right\},\ p \ge 1 \quad (24)$$

3. Yager union:

$$\mu_{\tilde{A}\cap\tilde{B}}(x) = \min\left\{1, \left(\mu_{\tilde{A}}(x)^p + \mu_{\tilde{B}}(x)^p\right)^{1/p}\right\},\ p \ge 1 \quad (25)$$

4. Dubois and Prade intersection:

$$\mu_{\tilde{A}\cap\tilde{B}}(x) = \frac{\mu_{\tilde{A}}(x)\cdot\mu_{\tilde{B}}(x)}{\max\left\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x), \alpha\right\}}, \alpha \in [0,1] \quad (26)$$

5. Union:

$$\mu_{\tilde{A}\cup\tilde{B}}(x) = \frac{\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x)\cdot\mu_{\tilde{B}}(x) - \min\left\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x), (1-\alpha)\right\}}{\max\left\{(1-\mu_{\tilde{A}}(x)), (1-\mu_{\tilde{B}}(x)), \alpha\right\}}. \quad (27)$$

Finally, a class of 'averaging operators'[29] were defined, which do not have the mathematical properties of the t-norms and t-conorms. A justification for those operators will be given in Section *Empirical Evidence*. Two of the best known are the following two:

Definition 13 *The compensatory and operator is defined as follows:*

$$\mu_{\tilde{A}_{i,\mathrm{comp}}}(x) = \left(\prod_{i=1}^{m}\mu_i(x)\right)^{(1-\gamma)}\left(1 - \prod_{i=1}^{m}(1-\mu_i(x))\right)^{\gamma}$$
$$x \in X, \le \gamma \le 1. \quad (28)$$

In effect, each convex combination of a t-norm with the respective t-conorm could be used as averaging operator.

Definition 14 *An OWA-Operator*[30] *is defined as follows:*

$$\mu_{\mathrm{OWA}}(x) = \sum_j w_j\mu_j(x) \quad (29)$$

where $w = \{w_1, \ldots, w_n\}$ *is a vector of weights* w_i *with* $w_i \in [0,1]$ *and* $\sum_i w_i = 1$.

$\mu_j(x)$ is the jth largest membership value for an element x for which the (aggregated) degree of membership shall be determined. The rationale behind this operator is again the observation that for an 'and' aggregation (modeled i.e., by the min-operator) the smallest degree of membership is crucial, whereas for an 'or' aggregation (modeled by 'max') the largest degree of membership of an element in all fuzzy sets is to be aggregated. Therefore, a basic aspect of this operator is the re-ordering step. In particular, the degree of membership of an element in a fuzzy set is not associated with a particular weight. Rather a weight is associated with a particular ordered position of a degree of membership in the ordered set of relevant degrees of membership (Table 1).

The Extension Principle

One of the most basic concepts of fuzzy set theory that can be used to generalize crisp mathematical concepts to fuzzy sets is the extension principle. In its elementary form, it was already implied in Zadeh's first contribution. In the meantime, modifications have been suggested. Following Zadeh and Dubois and Prade,[2,10] we define the extension principle as follows:

Definition 15 *Let* X *be a Cartesian product of universes* $X = X_1 \times \cdots \times X_r$, *and* $\tilde{A}_1, \ldots, \tilde{A}_r$ *be* r *fuzzy sets in* $X_1, \ldots, X_r$, *respectively.* f *is a mapping from* X *to a universe* Y, $y = f(x_1, \ldots, x_r)$. *Then the extension principle allows us to define a fuzzy set* $\tilde{B}$ *in* Y *by*

$$\tilde{B} = \left\{(y, \mu_{\tilde{B}}(y)) | y = f(x_1, \ldots, x_r), (x_1, \ldots, x_r) \in X\right\} \quad (30)$$

TABLE 1 | Classification of Aggregation Operators

References	Intersection operators t-norms	Averaging operators	Union operators t-conorms
		Nonparameterized	
Zadeh 1965	Minimum		Maximum
	Algebraic product		Algebraic sum
Giles 1976	Bounded sum		Bounded difference
Hamacher 1978	Hamacher product		Hamacher sum
Mizumoto 1982	Einstein product		Einstein sum
Dubois and Prade 1980,1982	Drastic product		Drastic sum
Dubois and Prade 1984		Arithmetic mean Geometric mean	
Silvert 1979		Symmetric summation	
		Symmetric difference	
		Parameterized families	
Hamacher 1978	Hamacher intersection operators		Hamacher union operators
Yager 1980	Yager intersection operators		Yager union operators
Dubois and Prade 1980, 1982, 1984	Dubois intersection operators		Dubois union operators
Werners 1984		'fuzzy and', 'fuzzy or'	
Zimmermann and Zysno 1984		'compensatory and', convex comb. of maximum and minimum , or algebraic product and algebraic sum	
Yager 1988		OWA operators	

where

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup_{(x_1,\ldots,x_r)\in f^{-1}(y)} \min\{\mu_{\tilde{A}_1}(x_1),\ldots,\mu_{\tilde{A}_r}(x_r)\} \\ \text{if } f^{-1}(y) \neq \Theta \\ 0 \quad \text{otherwise} \end{cases} \tag{31}$$

where f^{-1} is the inverse of f.

For $r = 1$, the extension principle, of course, reduces to

$$\tilde{B} = f(\tilde{A}) = \{(y, \mu_{\tilde{B}}(y)) | y = f(x), x \in X\} \tag{32}$$

where

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup_{x\in f^{-1}(y)} \min\{\mu_{\tilde{A}}(x)\} & \text{if } f^{-1}(y) \neq \Theta \\ 0 & \text{otherwise} \end{cases} \tag{33}$$

Fuzzy Relations and Graphs

Fuzzy relations are fuzzy sets in product space.[31,32]

Definition 16 *Let $X, Y \subseteq \mathbb{R}$ be universal sets, then*

$$\tilde{R} = \{((x, y), \mu_{\tilde{R}}(x, y)) | (x, y) \subseteq X \times Y\} \tag{34}$$

is called a fuzzy relation on $X \times Y$.

Example 1 *Let $X = Y = \mathbb{R}$ and $\tilde{R}$:= 'considerably larger than'.*
The membership function of the fuzzy relation, which is, of course, a fuzzy set on $X \times Y$ can then be

$$\mu_{\tilde{R}}(x, y) = \begin{cases} 0 & \text{for } x \le y \\ \frac{(x-y)}{10y} & \text{for } y < x \le 11y \\ 1 & \text{for } x > 11y \end{cases} \tag{35}$$

A different membership function for this relation could be

$$\mu_{\tilde{R}}(x, y) = \begin{cases} 0 & \text{for } x \leq y \\ (1 + (y - x)^{-2})^{-1} & \text{for } x > y. \end{cases} \tag{36}$$

For discrete supports, fuzzy relations can also be defined by matrices.[21] Quite a number of different kinds of fuzzy relations have been defined. As an example, we will here only define one kind: similarity relations[33]:

Definition 17 *A fuzzy relation that is reflexive, symmetric, and transitive is called a similarity relation.*

A fuzzy relation $\tilde{R}(x_0 x)$ is called

$$\begin{array}{ll} \text{reflexive} & \text{if } \mu_{\tilde{R}}(x, x) = 1 \\ \text{symmetric} & \text{if } \mu_{\tilde{R}}(x, y) = \mu_{\tilde{R}}(y, x) \end{array} \tag{37}$$

$$\left.\begin{array}{l} \max - \min \\ \text{transitive} \end{array}\right\} \text{if } \mu_{\tilde{R}\circ\tilde{R}} \leq \mu_{\tilde{R}}(x, y). \tag{38}$$

Definition 18 *Max-min composition: let $\tilde{R}_1(x, y)$, $(x, y) \in X \times Y$ and $\tilde{R}_2(y, z)$, $(y, z) \in Y \times Z$ be two fuzzy set relations. The max-min composition $\tilde{R}_1$ max − min $\tilde{R}_2$ is then the fuzzy set*

$$\tilde{R}_1 \circ \tilde{R}_2 = \left\{ \left[(x, z), \max_y \left\{ \min \left\{ \mu_{\tilde{R}_1}(x, y), \mu_{\tilde{R}_2}(y, z) \right\} \right\} \right] \middle| x \in X, y \in Y, z \in Z \right\}. \tag{39}$$

$\mu_{\tilde{R}_1 \circ \tilde{R}_2}$ is again the membership function of a fuzzy relation on fuzzy sets.

The 'counterpart' of similarity relations are order relations or preference relations[21] which are further differentiated in fuzzy preorders, fuzzy total orders, or fuzzy strict total orders, which are particularly often used in multicriteria analysis and preference theory. Fuzzy relations can also be considered as representing fuzzy graphs.[11] Let the elements of fuzzy relations, as defined in Definition 16 be the nodes of a fuzzy graph that is represented by this fuzzy relation. The degrees of membership of the elements of the related fuzzy sets define the 'strength' of or the flow in the respective nodes of the graph, whereas the degrees of membership of the corresponding pairs in the relation are the 'flows' or the 'capacities' of the edges. Fuzzy graph theory has in the meantime become an extended area of (fuzzy) mathematics.

Fuzzy Analysis

A fuzzy function is a generalization of the concept of a classical function. A classical function f is a mapping (correspondence) from the domain D of definition of the function into a space S; $f(D) \subseteq S$ is called the range of f. Different features of the classical concept of a function can be considered fuzzy rather than crisp. Therefore, different 'degrees' of fuzzification of the classical notion of a function are conceivable:

1. There can be a crisp mapping from a fuzzy set that carries along the fuzziness of the domain and, therefore, generates a fuzzy set. The image of a crisp argument would again be crisp.
2. The mapping itself can be fuzzy, thus blurring the image of a crisp argument. This is normally called a *fuzzy function*. Dubois and Prade call this 'fuzzifying function'.[10]
3. Ordinary functions can have properties or be constrained by fuzzy constraints.

Particularly for Case 2, definitions for classical notions of analysis, such as, extrema of fuzzy functions, integration of fuzzy functions, integration of fuzzy functions aver a crisp interval, integration of a crisp function over a fuzzy interval, fuzzy differentiation, and so on have been suggested.[9,10,34,35] It would exceed the scope of this survey to describe even a larger part of them in more detail.

Empirical Evidence

So far fuzzy sets and their extensions and operations were considered as formal concepts that need no proof by reality. If, however, fuzzy concepts are used, for example, to model human language, then one has to make sure that the concepts really model, what a human says or thinks. One has, with other words, to 'extract' thoughts from the human brain and compare them with the modeling tools and concepts. This is a problem of psycho-linguistics.[6,36] Bellman and Zadeh[37] suggested in their paper that the 'and' by which in a decision objective functions and constraints are combined can be modeled by the intersection of the respective fuzzy sets and mathematically modeled by 'min' or by the product.

The interpretation of a decision as the intersection of fuzzy sets implies no positive compensation (trade-off) between the degrees of membership of the fuzzy sets in question, if either the minimum or the product is used as an operator. Each of them yields degrees of membership of the resulting fuzzy set (decision) which are on or below the lowest degree of membership of all intersecting fuzzy sets. This is also true if any other t-norm is used to model the intersection or operators, which are no t-norms but map below the min-operator. It should be noted that, in fact, there are decision situations in which such a 'negative compensation' (i.e., mapping below the minimum) is appropriate. The interpretation of a decision as the union of fuzzy sets, using the max-operator, leads to the maximum degree of membership achieved by any of the fuzzy sets representing objectives or constraints. This amounts to a full compensation of lower degrees of membership by the maximum degree of membership. No membership will result, however, which is larger than the largest degree of membership of any of the fuzzy sets involved. Observing managerial decisions, one finds that there are hardly any decisions with no compensation between either different degrees of goal achievement or the degrees to which restrictions are limiting the scope of decisions. It may be argued that compensatory tendencies in human aggregation are responsible for the failure of some classical operators (min, product, max) in empirical investigations.[38]

The following conclusions can probably be drawn: neither t-norms nor t-conorms can alone cover the scope of human aggregating behavior. It is very unlikely that a single nonparametric operator can model appropriately the meaning of 'and' or 'or' context independently, that is, for all persons, at any time and in each context. There seem to be three ways to remedy this weakness of t-norms and t-conorms: one can either define parameter-dependent t-norms or t-conorms that cover with their parameters the scope of some of the nonparametric norms and can, therefore, be adapted to the context. A second way is to combine t-norms and their respective t-conorms and such cover also the range between t-norms and t-conorms (which may be called the range of partial positive compensation). The disadvantage is that generally some of the useful properties of t- and s-norms get lost. The third way is, eventually, to design operators, which are neither t-norms nor t-conorms, but which model specific contexts well. Empirical validations of suggested operators are very scarce. The 'min', 'product', 'geometric mean', and the γ-operator have been tested empirically and it has turned out, the γ-operator models the 'linguistic and', which lies between[36,39] the 'logical and' and the 'logical exclusive or', best and context dependently. It is the convex combination of the product (as a t-norm) and the generalized algebraic sum (as a t-conorm). In addition, it could also be shown that this operator is pointwise injective, continuous, monotonous, commutative and in accordance with the truth tables of dual logic. Limited empirical tests have also been executed for shapes of membership functions, linguistic approximation, and hedges.[40–45]

APPLICATIONS

It shall be stressed that 'applications' in this review mean applications of fuzzy set theory to other formal theories or techniques and not to real problems. The latter would certainly exceed the scope of this review. The interested reader is referred to Refs 46–54.

Fuzzy Logic, Approximate Reasoning, and Plausible Reasoning

Logics as bases for reasoning can be distinguished essentially by three topic-neutral items: truth values, vocabulary (operators), and reasoning procedures (tautologies, syllogisms). In dual logic, truth values can be'true' (1) or 'false' (0) and operators are defined via truth tables (e.g., Table 2).

A and *B* represent two sentences or statements which can be true (1) or false (0). These statements can be combined by operators. The truth values of the combined statements are shown in the columns under the respective operators 'and', 'inclusive or', 'exclusive or', implication, and so on. Hence, the truth values in the columns define the respective operators. Considering the *modus ponens* as one tautology:

$$(A \wedge (A \Rightarrow B)) \Rightarrow B \tag{40}$$

or:

Premise: *A* is true
Implication: If *A* then *B*
Conclusion: *B* is true.

Here four assumptions are being made:

1. *A* and *B* are crisp.
2. *A* in premise is identical to *A* in implication.
3. True = absolutely true
   False = absolutely false.
4. There exist only two quantifiers: 'All' and 'There exists at least one case'.

TABLE 2 | Truth Table of Dual Logic

A	*B*	$\wedge$	$\vee$	$x\vee$	$\Rightarrow$	$\Leftrightarrow$	?
1	1	1	1	0	1	1	1
1	0	0	1	1	0	0	1
0	1	0	1	1	1	0	0
0	0	0	0	0	1	1	0

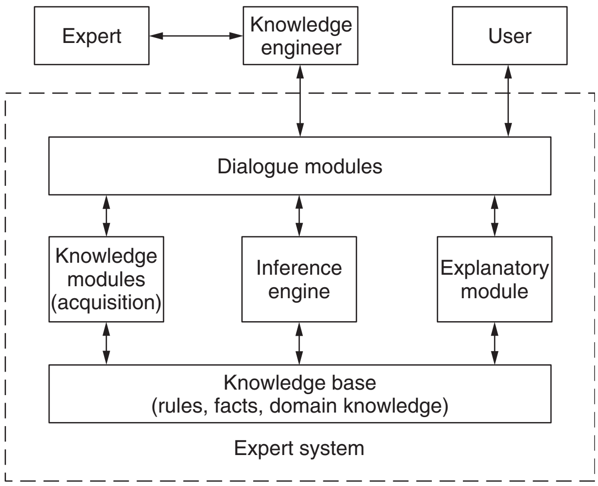

FIGURE 3 | Expert system.

These assumptions are relaxed in fuzzy reasoning.[55–61]

In *fuzzy logic*, the truth values are no longer restricted to the two values 'true' and 'false' but are expressed by the linguistic variables 'true' and 'false'. In *approximate reasoning*[62] additionally the statements A and/or B can be fuzzy sets. In *plausible reasoning*,[21] the A in the premise does not have to be identical (but similar) to the A in the implication. In all forms of fuzzy reasoning, the implications can be modeled in many different ways.[60] Which one is the most appropriate can be evaluated either empirically[36] or axiomatically.[60] Of course, the models for implications can also be chosen with respect to their computational efficiency (which does not guaranty that the proper model has been chosen for a certain context).

Fuzzy Rule-Based Systems (Fuzzy Expert Systems and Fuzzy Control)

Knowledge-based systems are computer-based systems, normally to support decisions in which mathematical algorithms[63] are replaced by a knowledge base and an inference engine. The knowledge base[64] contains expert knowledge. There are different ways to acquire and store expert knowledge.[65,66] The most frequently used way to store this knowledge are if–then rules. These are then considered as 'logical' statements, which are processed in the inference engine to derive a conclusion or decision. Generally, these systems are called 'expert systems' (Figure 3). Classical expert systems processed the truth values of the statements. Hence, they were actually not processing knowledge but symbols.

In the 1970s, crisp rules in the knowledge base were substituted by fuzzy statements, which semantically contained the context of the rules.[67] Naturally the inference engine had to be substituted by a system that was able to infer from fuzzy statements. It shall be called here 'computational unit'. The first, very successful, applications of these systems were in control engineering. They were, therefore, called 'fuzzy controllers'.[20,68–71] The input to these systems was numerical (measures of the process output) and had to be transformed into fuzzy statements (fuzzy sets), which was called 'fuzzification'. This input, together with the fuzzy statements of the rule base, was then processed in the computational unit, which delivered again fuzzy sets as output. Because the output was to control processes, it had to be transformed again into real numbers. This process was called 'defuzzification'[72] (Figure 4). This is one difference to fuzzy expert systems, which are supposed to replace or support human experts. Hence, the output should be linguistic and, rather than having 'defuzzification' at the end, these systems use 'linguistic approximation'[73] to provide user-friendly output.

Several methods have been developed for fuzzification, inference, and defuzzification, and at the beginning of the 1980s the first commercial systems were put on the market (control of video cameras, cement kilns, cameras, washing machines, etc.). This development started primarily in Japan and then spread to Europe and the United States. It boosted the interest in fuzzy set theory tremendously, such that in Europe one was talking of a 'fuzzy boom' around 1990. Research and development in the area of fuzzy control is, however, still ongoing to day.

Fuzzy Data Mining

With the development of electronic data processing, more and more data were available electronically. This led to the situation in which the masses of data, for instance in data warehouses, exceeded the human capabilities to recognize important structures in these data. Classical methods to 'data mine', such as cluster techniques, and so on, were available, but often they did not match the needs. Cluster techniques, for instance, assumed that data could be subdivided crisply into clusters, which did not fit the structures that existed in reality. Fuzzy set theory seemed to offer good opportunities to improve existing concepts. Bezdek[31,74] was one of the first, who developed fuzzy cluster methods with the goals, to search for structure in data to reduce complexity and to provide input for control and decision making (Figure 5).

Different approaches have been followed: hierarchical approaches, semi-formal heuristic approaches, and objective function clustering. Because the area

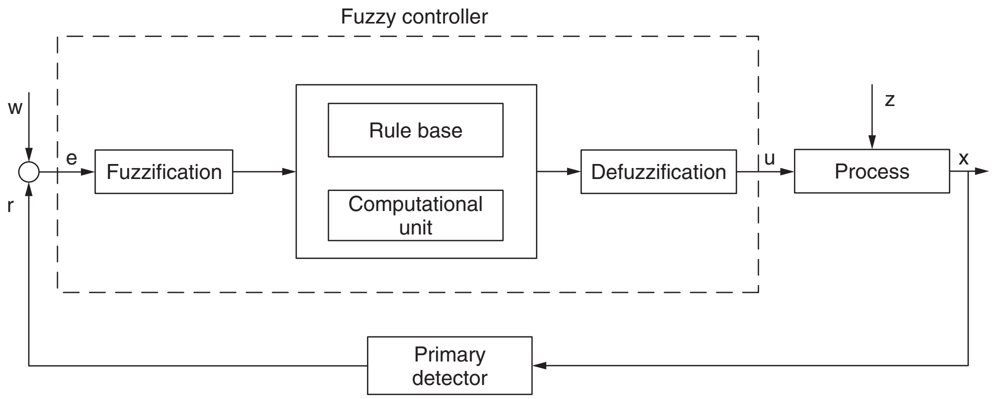

FIGURE 4 | Fuzzy controller.

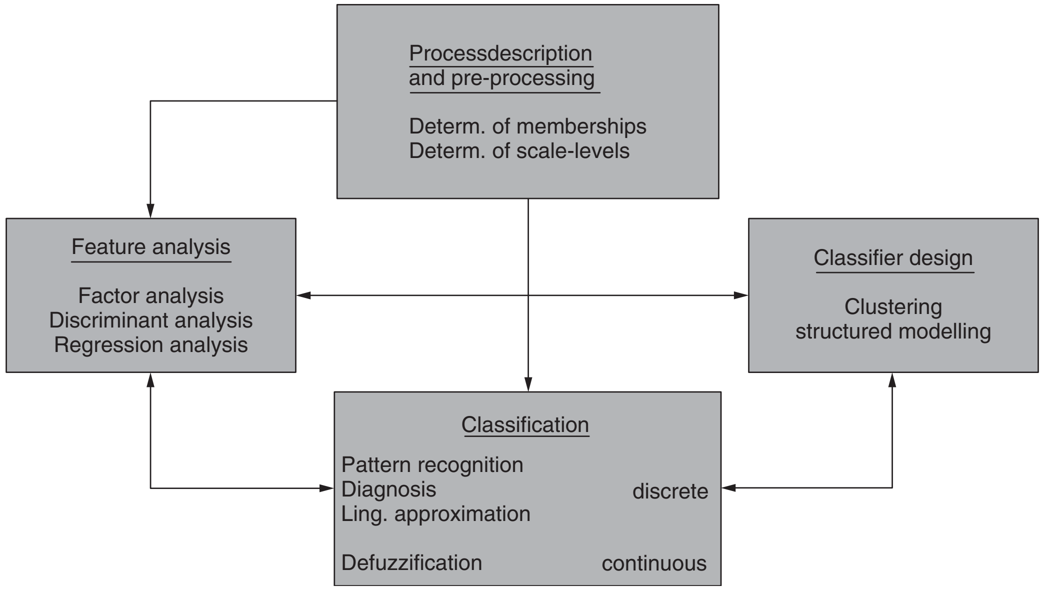

FIGURE 5 | Fuzzy data mining.

of (intelligent) data mining becomes more and more important, the development of further fuzzy methods in this area can be expected.[46,75–82]

Fuzzy Decisions

In the logic of decision making, a decision is defined as the choice of a (feasible) action by which the utility function is maximized. Hence, it is the search for an optimal *and* feasible action or strategy. Feasibility is either defined by enumerating the feasible actions or by constraints that define the feasibility space. The feasibility space is unordered with respect to utility and the utility or objective function defines an order in this space. Hence, the problem is not symmetric. The problem becomes more complicated if several objective functions exist (several orders on the same space). This leads to the area of 'multicriteria analysis'. This area has grown very much since the 1970s. Many approaches have been suggested to solve problems with several objective functions. In all these approaches, objective functions were considered to be real valued and the actions as crisply defined. For the case that the objective function or the constraints are not crisply defined Bellman and Zadeh suggested in 1970 the following symmetric model:[37]

Definition 19 *Let $\mu_{\tilde{C}_i}, i = 1, \ldots, m$ be membership functions of constraints on X, defining the decision space and $\mu_{\tilde{G}_j}, j = 1, \ldots, n$ the membership functions of objective (utility) functions or goals on X.*

A decision is then defined by its membership function

$$\begin{aligned}\mu_{\tilde{D}} &= (\mu_{\tilde{C}_1} * \cdots * \mu_{\tilde{C}_m}) \times (\mu_{\tilde{G}_1} * \cdots * \mu_{\tilde{G}_n}) \\ &= *_i \mu_{\tilde{C}_i} \times *_j \mu_{\tilde{G}_j} \end{aligned} \quad (41)$$

where $\times, *$ denote appropriate, possibly context dependent, 'aggregators' (connectives).

The most frequent interpretation of $\times, *$ is the min-operator.

It should be clear that the intersection of the objective functions and the constraints (called 'decision') is again a fuzzy set. If a crisp decision is needed, one could, for instance, determine the solution that has the highest degree of membership in this fuzzy set. Let us call this solution the 'maximizing solution'.

This definition complicated and facilitated the classical model at the same time: it facilitated the model by suggesting a symmetric rather than an asymmetric model. It complicated the problem because now to determine the optimal action, not real numbers (of the real-valued utility function) but functions (the membership functions of fuzzy numbers) had to be compared and ranked. The later problem led to many publications that focused on the ranking of fuzzy numbers.[83,84] The symmetry of the model led to very efficient approaches in the area of 'multiobjective decision making', that is, in multiobjective mathematical programming, in which several continuous objective functions have to be optimized subject to constraints. In fact, mathematical programming can be considered as a special case of decisions in the logic of decision making. This will be considered in more detail in the next section. For fuzzy multistage decisions, see Refs 85,86.

Fuzzy Optimization

Fuzzy optimization[87] was already discussed briefly in Section *Fuzzy Analysis*. Here, we shall concentrate on 'constrained optimization', which is generally called 'mathematical programming'.[88–95] It will show the potential of the application of fuzzy set theory to classical techniques very clearly. We shall look at the easiest, furthest developed, and most frequently used version, the linear programming. It can be defined as follows:

Definition 20 *Linear programming model:*

$$\begin{gathered} \max\ f(x) = z = c^T x \\ \text{s.t. } Ax \leq b \\ x \geq 0 \end{gathered}$$

with $c, x \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m\times n}$.

In this model it is normally assumed that all coefficients of A, b, and c are real (crisp) numbers, that '$\leq$' is meant in a crisp sense, and that 'maximize' is a strict imperative. This also implies that the violation of any single constraint renders the solution unfeasible and that all constraints are of equal importance (weight). Strictly speaking, these are rather unrealistic assumptions, which are partly relaxed in *fuzzy linear programming*.[96–100]

If we assume that the LP-decision has to be made in fuzzy environments, quite a number of possible modifications exist. First of all, the decision maker might really not want to actually maximize or minimize the objective function. Rather he might want to reach some aspiration levels which might not even be definable crisply. Thus, he might want to 'improve the present cost situation considerably', and so on. Second, the constraints might be vague in one of the following ways: the '$\leq$' sign might not be meant in the strictly mathematical sense but smaller violations might well be acceptable. This can happen if the constraints represent aspiration levels as mentioned above or if, for instance, the constraints represent sensory requirements (taste, color, smell, etc.) which cannot adequately be approximated by a crisp constraint. Of course, the coefficients of the vectors b or c or of the matrix A itself can have a fuzzy character either because they are fuzzy in nature or because perception of them is fuzzy. Finally, the role of the constraints can be different from that in classical linear programming where all constraints are of equal weight. For the decision maker, constraints might be of different importance or possible violations of different constraints may be acceptable to him to different degrees. Fuzzy linear programming offers a number of ways to allow for all those types of vagueness and we shall discuss some of them below. If we assume that the decision maker can establish an aspiration level, z, of the objective function, which he wants to achieve as far as possible and if the constraints of this model can be slightly violated—without causing unfeasibility of the solution—then the model can be written as follows:

$$\begin{gathered} \text{Find } x \\ \text{s.t. } c^T x \tilde{\geq} z \\ Ax \tilde{\leq} b \\ x \tilde{\geq} 0. \end{gathered} \tag{42}$$

Here, $\tilde{\leq}$ denotes the fuzzified version of $\leq$ and has the linguistic interpretation 'essentially smaller than or equal'. $\tilde{\geq}$ denotes the fuzzified version of $\geq$ and has the linguistic interpretation 'essentially greater than or equal'. The objective function might have to be written as a minimizing goal to consider z as an upper bound. Obviously the asymmetric linear programming model has now been transformed into a symmetric model. To make this even more visible, we shall rewrite the model as

$$\begin{gathered} \text{Find } x \\ \text{s.t. } Bx \tilde{\leq} d \\ x \geq 0. \end{gathered} \tag{43}$$

The fuzzy set 'decision' $\tilde{D}$ is then

$$\mu_{\tilde{D}}(x) = \min_{i=1,\ldots,m+1} \{\mu_i(x)\}. \tag{44}$$

$\mu_i(x)$ can be interpreted as the degree to which x fulfills (satisfies) the fuzzy inequality $B_i x \tilde{\leq} d_i$ (where B_i denotes the ith row of B). Assuming that the decision maker is interested not in a fuzzy set but in a crisp 'maximizing' solution, we could suggest the 'maximizing solution', which is the solution to the possibly nonlinear programming problem

$$\max_{x \geq 0} \min_{i=1,\ldots,m+1} \{\mu_i(x)\} = \max_{x \geq 0} \mu_{\tilde{D}}(x). \tag{45}$$

As membership functions seem suitable.

$$\mu_i(x) = \begin{cases} 1 & \text{if } B_i x \leq d_i \\ 1 - \frac{B_i x - d_i}{p_i} & \text{if } d_i < B_i x \leq d_i + p_i, \\ & \quad i = 1, \ldots, m+1 \\ 0 & \text{if } B_i x > d_i + p_i. \end{cases} \tag{46}$$

The p_i are subjectively chosen constants of admissible violations of the constraints and the objective function. Substituting these membership functions into the model yields, after some rearrangements and with some additional assumptions

$$\max_{x \geq 0} \min_{i=1,\ldots,m+1} \left\{ 1 - \frac{B_i x - d_i}{p_i} \right\}. \tag{47}$$

Introducing one new variable, λ, which corresponds essentially to the degree of membership of x in the fuzzy set decision, we arrive at

$$\begin{gathered} \max \lambda \\ \text{s.t. } \lambda p_i + B_i(x) \leq d_i + p_i, \; i = 1, \ldots, m+1 \\ x \geq 0. \end{gathered} \tag{48}$$

A slightly modified version of this model results if the membership functions are defined as follows: a variable t_i, $i = 1, \ldots, m+1$, $0 \leq t_i \leq p_i$ is defined which measures the degree of violation of the ith constraint. The membership function of the ith row is then

$$\mu_i(x) = 1 - \frac{t_i}{p_i}. \tag{49}$$

The crisp equivalent model is then

$$\begin{gathered} \max \lambda \\ \text{s.t. } \lambda p_i + t_i \leq p_i \\ B_i x - t_i \leq d_i, \; i = 1, \ldots, m+1 \\ t_i \leq p_i \\ x, t \geq 0. \end{gathered} \tag{50}$$

This is a normal crisp linear programming model for which very efficient solution methods exist.

The main advantage, compared with the unfuzzy problem formulation, is the fact that the decision maker is not forced into a precise formulation because of mathematical reasons although he might only be able or willing to describe his problem in fuzzy terms. Linear membership functions are obviously only a very rough approximation. Membership functions which monotonically increase or decrease, respectively, in the interval of $[d_i, d_i + p_i]$ can also be handled quite easily. It should also be observed that the classical assumption of equal importance of constraints has been relaxed: the slope of the membership functions determines the 'weight' or importance of the constraint. The slopes, however, are determined by the p_is. The smaller the p_i the higher the importance of the constraint. For $p_i = 0$, the constraint becomes crisp, that is, no violation is allowed. Because of the symmetry of the model, it is very easy to add additional objective functions, which solves the problem of multiobjective decision making.[101–107] Because this is a crisp model, existing crisp constraints can also be added easily. So far, two major assumptions have been made to arrive at 'equivalent models' which can be solved efficiently by standard LP-methods:

1. Linear membership functions were assumed for all fuzzy sets involved.
2. The use of the minimum-operator for the aggregation of fuzzy sets was considered to be adequate.

re1 The linear membership functions used so far could all be defined by fixing two points, the upper and lower aspiration levels or the two bounds of the tolerance interval. The obvious way to handle nonlinear membership functions is probably to approximate them piecewise by linear functions.[108,109]

re2 As already mentioned earlier, the min-operator models not always the real meaning of the 'linguistic and'. Hence, the modeler might want to use other operators. The computational efficiency, by which the problem can then be solved, depends only on the type of crisp equivalent model, for example, whether it is linear or nonlinear, and that depends on the combination of the type of membership function and the operator used in the fuzzy model. Table 3 shows these possible combinations and the type of resulting crisp equivalent model.

TABLE 3 | Resulting Equivalent Models

Membership function	Operator	Model
Linear	min	LP
Logistic	min	LP
Hyperbolic	min	LP
Linear	γ min $+(1-\gamma)$ max	MILP
Linear/nonlinear	Product	nonconvex NLP
Linear/nonlinear	γ-operator	nonconvex NLP
Linear	$\widetilde{and}$	LP
Linear	$\widetilde{or}$	MILP
Logistic	$\widetilde{and}$	NLP with lin. const.

With $\widetilde{and} = \gamma \min(x, y) + (1-\gamma)\frac{1}{2}(x+y)$ and $\widetilde{or} = \gamma \max(x, y) + (1-\gamma)\frac{1}{2}(x+y)$.

CONCLUSION

Since its inception in 1965 as a generalization of dual logic and/or classical set theory, fuzzy set theory has been advanced to a powerful mathematical theory. In more than 30,000 publications, it has been applied to many mathematical areas, such as algebra, analysis, clustering, control theory, graph theory, measure theory, optimization, operations research, topology, and so on. In addition, alone or in combination with classical approaches it has been applied in practice in various disciplines, such as control, data processing, decision support, engineering, management, logistics, medicine, and others. It is particularly well suited as a 'bridge' between natural language and formal models and for the modeling of nonstochastic uncertainties.

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