YLE13: Hotelling model

Marko Lindroos

Questions to be answered wrt non-renewable resources

- What is the optimal extraction rate q?
- Market prices p in time?
- When do we run out $x(T)=0$, T?

Answers will depend on

- Demand
- Discount rate
- Known reserves of the resource
- Price of the subsitute

Hotelling model (JPE 1931)

- Initial stock size is x(0), the resource stock decreases in time when it is used.

(1) $$x(t) = x(0) - \int_0^t q(t)\,dt$$

Equation of motion

$$\dot{x}(t) = -q(t)$$

- The resource is used untill it is exhausted at time T

$$\int_0^T q(t)\,dt \le x(0)$$

Competitive market

Objective function

- Maximise the Net Present Value by choosing the extraction q(t)

- Max J= $\int_0^T e^{-rt} q(t)(p(t) - c)\,dt$

- St equation of motion
- c = (constant) unit cost of extraction

Optimal control problem

- $q(t)$ — control variable
- $x(t)$ — state variable
- $\dot{x}(t) = -q(t)$ — equation of motion
- $x(0)$ — initial state

Current value Hamiltonian

$$H = q(t)(p(t) - c) - m(t)q(t)$$

Maximun principle

$$\frac{\partial H}{\partial q} = 0$$

$$-\frac{\partial H}{\partial x} = \dot{\mu}(t) - r\mu(t)$$

FOC

$$\frac{\partial H}{\partial q} = 0 \Rightarrow \quad p(t) - c = \mu(t)$$

Interpretation

- Net revenue= scarcity price of the resource

Comparison to regular market

$$p(t) = MC + m(t)$$

- Non-renewable resource price is higher than ”normal” competitive market.
- Scarcity price measures the difference.

Dynamic condition

$$-\frac{\partial H}{\partial x} = \dot{\mu}(t) - \rho\mu(t)$$

$$\Rightarrow \dot{\mu}(t) - \rho\mu(t) = 0$$

Hotelling rule

$$\frac{\dot{m}(t)}{m(t)} = r$$

Interpretation

- Scarcity price increases according to the discount rate. The resource should yield the same rate of interest than any other (risk-free) investment

1.3 Scarcity price in time

- Let us solve the differential equation (Hotelling’s rule)

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- On the LHS the time derivative of the discounted scarcity price

$$e^{-rt}\dot{m}(t) - e^{-rt} r m(t) = 0$$

- integrate

$$e^{-rt} m(t) = m(0)$$

…

$$m(t) = m(0)e^{rt}$$

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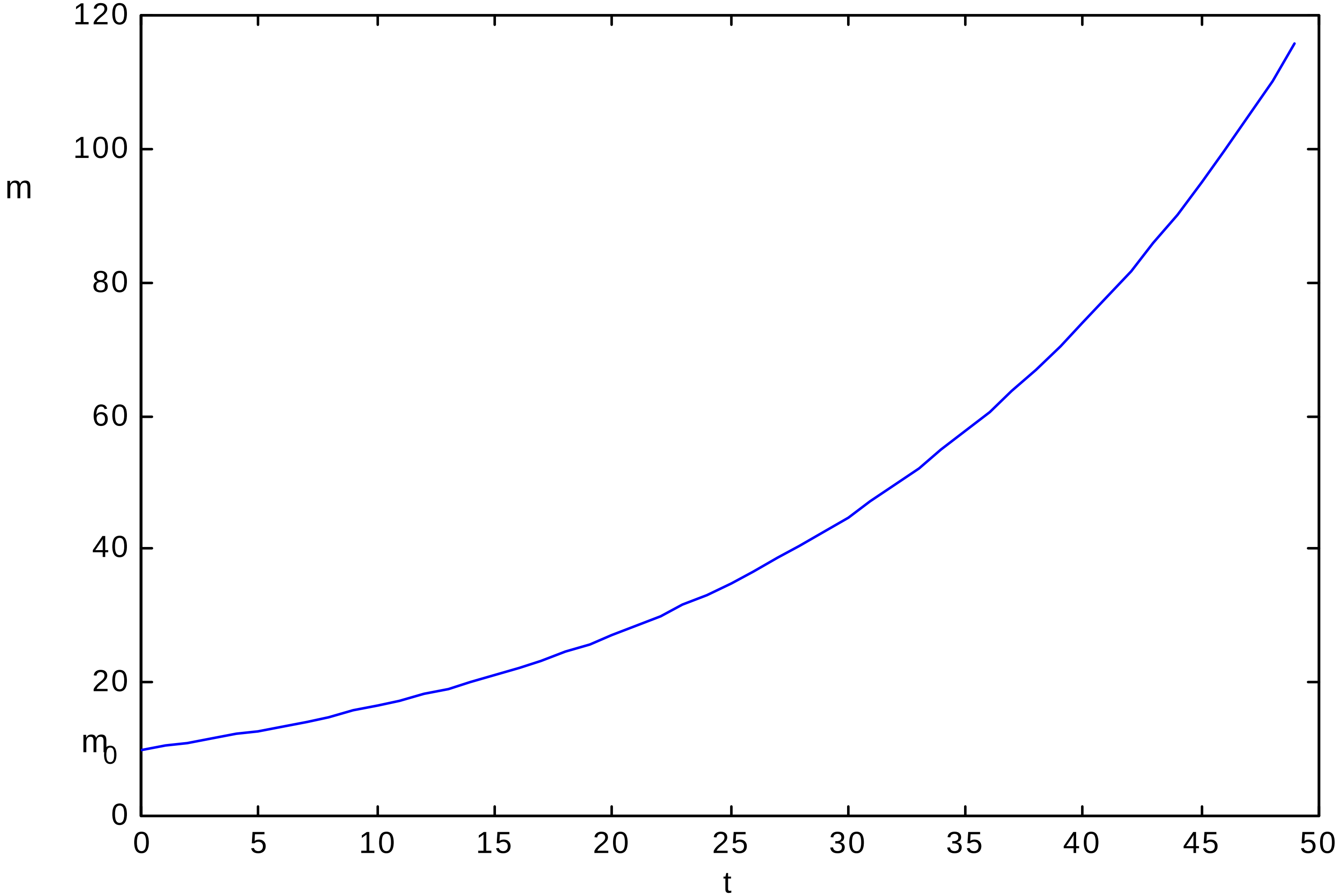
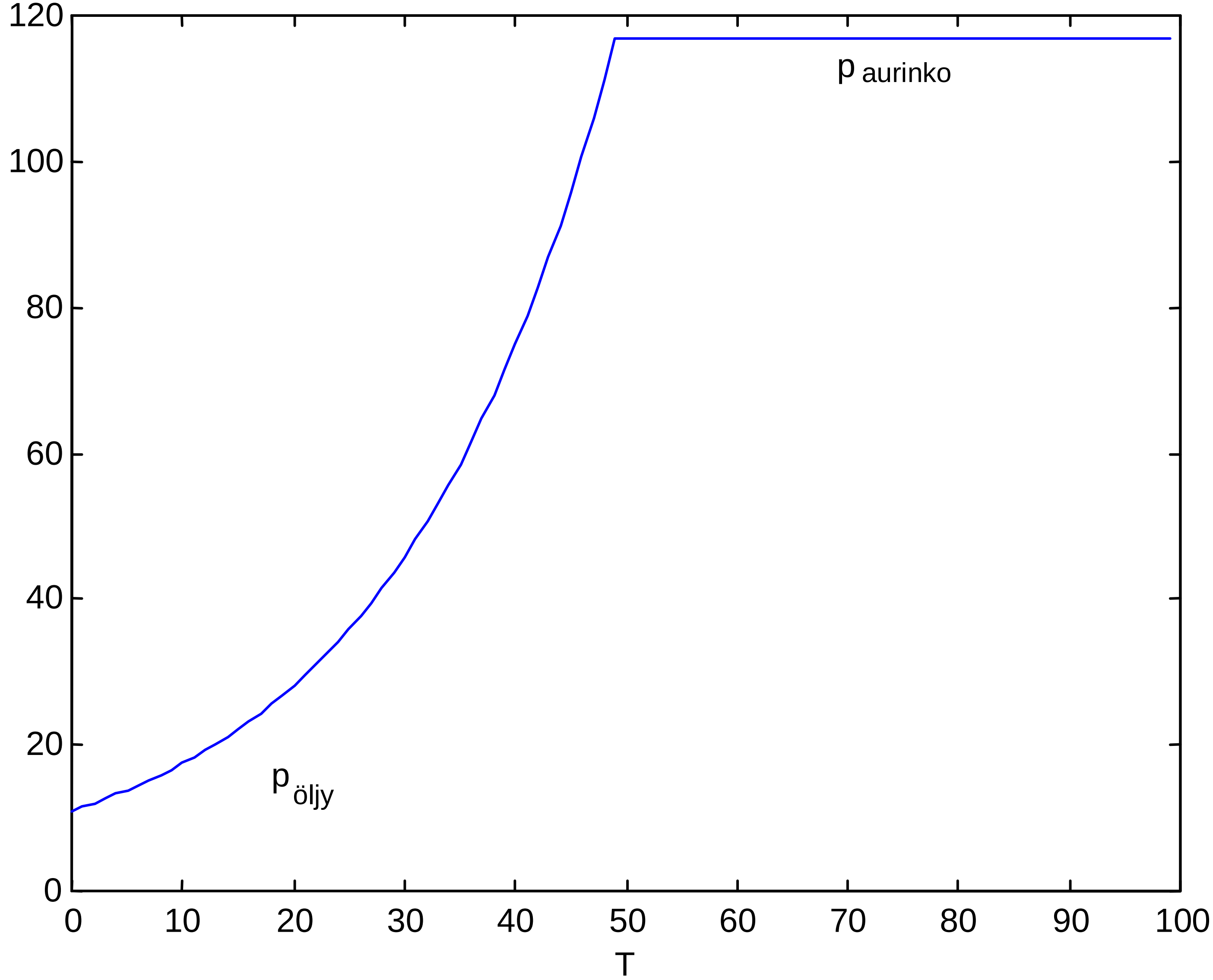
$$p(t) = m(0)e^{rt} + c$$

Timepath of scarcity (shadow) price
(m_0 = 10; r = 0.05;)



Backstop-price and optimal price of the resource

- Assume c=0.



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Initial price

- At T, price of the resource is equal to the backstop-price.

$$p(T) = \mu(T) = p^b$$

- At t=0 price can be computed since price increases now with the rate of discount.

$$p(0) = \mu(0) = p^b e^{-rT}$$

Optimal price

$$p_c(t) = p^b e^{r(t-T)}$$

1.5 Optimal rate of extraction and the optimal time to exhaustion

- Assume the following demand:

$$q(t) = \frac{p^b}{b} - \frac{p(t)}{b}$$

- The whole resource stock is exhausted

$$\int_0^{T_c} \left(\frac{p^b}{b} - \frac{p_c(t)}{b}\right) dt = x(0)$$

Optimal extraction rate

$$\int_0^{T_c} \left(\frac{p^b}{b} - \frac{p^b e^{r(t-T_c)}}{b} \right) dt = x(0)$$

Integrate to yield the time to exhaustion

$$\int_0^{T_c} \left(\frac{p^b}{b} t - \frac{p^b e^{r(t-T_c)}}{rb} \right) dt = x(0)$$

$$\frac{p^b}{b} T - \frac{p^b e^{r(T_c - T_c)}}{rb} - \left(0 - \frac{p^b e^{-rT_c}}{rb}\right) = x(0)$$

$$\boxed{\frac{p^b}{b} T - \frac{p^b}{rb} + \frac{p^b e^{-rT_c}}{rb} = x(0)}$$

Solving the time to exhaustion

- Needs to be computed numerically from the previous equation. It will be affected by backstop-price, discount rate, initial stock and demand.

- Note that this needs to be computed first before moving on to computing the optimal extraction and optimal price