YLE13: Optimal control theory

Marko Lindroos

Courses after

- Dynamic optimisation
- Licentiate course
- Numerical modeling course

Journals

- Natural Resource Modeling
- Marine Resource Economics
- Journal of Bioeconomics
- Resource and Energy Economics

Conferences

- Resource Modeling Association
- IIFET
- EAFE

Assumptions

- state of the natural resource changes in time (deterministic)
- objective is to optimise the use of the resource in the long-run
- state of the resource is a constraint in the economic optimisation problem

Optimal control

- **Optimal control problem:** How to choose the control to maximise economic benefits from the resource

- Optimal control results in an optimal time trajectory of the state

à

- We can define optimal use of natural resource for future years and the following resource stock level
- If we deviate from the optimal control at any point in time the economic benefits (Net Present Value) will decrease

Present value maximum principle

- Equation of motion for resource x:

$$\frac{dx}{dt} = f\left[x(t), u(t)\right], \qquad 0 \pounds t \pounds T$$

- Differential equation where state variable is x, control variable is u (for non-renewable we denote the control with q and renewable with h)

Initial and terminal stated

- Initial state x(0) known
- Terminal state x(T) is zero with non-renewable resources (in our Hotelling model)

Objective function

- Max $$J\{u(t)\} = \int_0^T e^{-rt} g[x(t), u(t)]\,dt$$

- St equation of motion, initial state, terminal state

Hamiltonian

$$H = e^{-rt} g\left[x(t), u(t)\right] + \lambda(t) f\left[x(t), u(t)\right]$$

- Here lambda is the costate variable. (Note that Lagrangian in static optimisation is very similar).

Hamiltonian measures:

- Instant benefits+ future benefits

Necessary conditions of the maximum principle

1. Optimal control

$$\boxed{\frac{\partial H}{\partial u(t)} = 0}$$

2. Dynamic condition

$$\boxed{\frac{d\lambda}{dt} = \dot{\lambda}(t) = -\frac{\partial H}{\partial x(t)}}$$

3. Equation of motion

$$\boxed{\frac{dx}{dt} = \dot{x}(t) = \frac{\partial H}{\partial \lambda(t)}}$$

Current value Hamiltonian

$$H_C = g\left[x(t), u(t)\right] + \mu(t)\, f\left[x(t), u(t)\right]$$

$$H_C = e^{rt} H_P$$

$$\mu(t) = e^{rt} \lambda(t)$$

- Myy = current value costate variable

Current value maximum principle

- Dynamic condition changes:

$$\dot{\lambda}(t) = -\frac{\partial H_P}{\partial x(t)} = -e^{-\rho t}\frac{\partial H_C}{\partial x(t)}$$

$$\dot{\lambda}(t) = e^{-\rho t}\dot{\mu}(t) - \rho e^{-\rho t}\mu(t)$$

$$-e^{-\rho t}\frac{\partial H_C}{\partial x(t)} = e^{-\rho t}\dot{\mu}(t) - \rho e^{-\rho t}\mu(t)$$

$$\boxed{-\frac{\partial H_C}{\partial x(t)} = \dot{\mu}(t) - \rho\mu(t)}$$

Maximum principle (current value)

1. Optimal control

$$\boxed{\frac{\partial H}{\partial u(t)} = 0}$$

2. Dynamic condition

$$\boxed{-\frac{\partial H_C}{\partial x(t)} = \dot{\mu}(t) - r\mu(t)}$$

3. Equation of motion

$$\boxed{\frac{dx}{dt} = \dot{x}(t) = \frac{\partial H}{\partial \mu(t)}}$$

Projects

- BIREGAME, game theory, valuation, Baltic Sea Fisheries (Univ Algarve, Univ Southern Denmark, Imperial College London, VATT)
- ECA, Economics of Aquatic foodwebs (VATT, SYKE, LUKE)
- NORMER, Climate change effects on marine ecosystems and natural resource economics (Univ Oslo coordinates)