2 x 2 matrix games, general treatment

Denote the two pure strategies of the game by R_1 and R_2, respectively. The pay-off matrix is

	R_1	R_2
R_1	a	b
R_2	c	d

R_1 is an ESS if $a>c$, because in this case a rare mutant with strategy R_2 in a population of R_1 individuals does worse than R_1 itself, and hence is not able to spread (invade). Similarly, R_2 is an ESS if $d>b$. Note that R_1 and R_2 may be ESS at the same time (if both $a>c$ and $d>b$ hold). In this case whichever of the two strategies is present in the beginning, it is resistant to invasion by the other.

Next, we look for mixed ESSs. Assume that the entire population uses a mixed strategy I, which applies R_1 with probability p and R_2 with probability 1-p. In this population, the pay-off of the pure strategy R_1 is given by

$$E(R_1,I) = p\,a + (1\text{-}p)\,b$$

because it encounters R_1 with probability p in which case it obtains a as pay-off, and it encounters R_2 with probability 1-p in which case it gets b. Similarly, the pay-off of R_2 is

$$E(R_2,I) = p\,c + (1\text{-}p)\,d$$

By the Bishop-Cannings theorem, these two pay-offs must equal with one another (and also with $E(I,I)$) if I is to be an ESS. Hence we can calculate p from the equation

$$p\,a + (1\text{-}p)\,b = p\,c + (1\text{-}p)\,d$$

and we get $p = \dfrac{d-b}{a-b-c+d}$ as candidate ESS.

In order to conclude that we have found a mixed ESS, we need to see (1) whether p is between 0 and 1 (if p is not a probability, this is not a meaningful strategy), and (2) whether the second ESS condition is satisfied. We check the latter point first. The second ESS condition requires that $E(I,J) > E(J,J)$ for any possible strategy J that differs from I. Denote the mixing frequency of the alternative strategy J by q. We are interested in any q that is not equal to p; pure strategies are also included as alternative strategies because q may be 0 (=pure R_2) or 1 (=pure R_1). We fill in the pay-offs from the matrix to obtain

$$E(I,J) = p\,q\,a + p\,(1\text{-}q)\,b + (1\text{-}p)\,q\,c + (1\text{-}p)\,(1\text{-}q)\,d$$

For example, the probability that strategy I uses R_1 is p; the probability that it meets a strategy-J individual who also uses R_1 is q; if both conditions hold true (with combined probability $p\ q$), then the I-individual gets a as pay-off. The other three terms of the above equation are interpreted analogously. In a similar way, we get

$$E(J,J) = q^2\ a + q\ (1\text{-}q)\ b + (1\text{-}q)\ q\ c + (1\text{-}q)^2\ d$$

To see whether the second ESS condition $E(I,J) > E(J,J)$ holds, let us compute the difference $E(I,J)$ - $E(J,J)$:

$$E(I,J) - E(J,J) =$$

$$= pqa + p(1\text{-}q)b + (1\text{-}p)qc + (1\text{-}p)(1\text{-}q)d - [q^2a + q(1\text{-}q)b + (1\text{-}q)qc + (1\text{-}q)^2d]$$

$$= (p\text{-}q)\ [\ q\ (a\text{-}b\text{-}c\text{+}d) + b - d\ ]$$

If I is indeed ESS, this last expression should be positive for every q that is different from p. Since we have $p = \dfrac{d-b}{a-b-c+d}$, what is in the square brackets can be written as $[(a\text{-}b\text{-}c\text{+}d)\ (q - p)]$, and we finally get

$$E(I,J) - E(J,J) = -\ (a\text{-}b\text{-}c\text{+}d)\ (p - q)^2$$

Since the square is always positive if p and q are different, the whole expression is positive, and hence the second ESS condition is satisfied, if $a\text{-}b\text{-}c\text{+}d$ is negative.

Finally, let's see whether $p = \dfrac{d-b}{a-b-c+d}$ is between 0 and 1 given that $a\text{-}b\text{-}c\text{+}d$ is negative. For p to be positive, d must be smaller than b, because the denominator is negative. For p to be less than 1, a must be smaller than c. We thus have a mixed ESS if both $a<c$ and $d<b$; in this case, $a\text{-}b\text{-}c\text{+}d$ is negative.

We can summarize the general conclusions about 2x2 matrix games as follows:

(i) R_1 is an ESS if $a>c$
(ii) R_2 is an ESS if $d>b$
(iii) There is a single mixed ESS if $a<c$ and $d<b$. (There was only one solution for p.)
The ESS mixing frequency is $p = \dfrac{d-b}{a-b-c+d}$.

(iv) The game always has an ESS (there is no combination of a, b, c, d that is not covered in the previous conditions).
(v) The game may have two pure ESSs at the same time, but if there is a mixed ESS, then no pure strategy is ESS.