HOW LARGE ISOTOPY IS NEEDED TO CONNECT HOMOTOPIC DIFFEOMORPHISMS (OF $\mathbb{T}^2$)

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Abstract. Given a diffeomorphism which is homotopic to the identity from the 2-torus to itself, we construct an isotopy whose norm is controlled by that of the diffeomorphism in question.

1. Introduction

Given a diffeomorphism which is homotopic to the identity from a closed surface to itself, it is known that one can improve the homotopy to an isotopy in a topological way ([5]) and further improve the differentiability of the isotopy ([3]), but the known procedures are subtle, causing the norms of the isotopy being difficult to control. In this paper, we introduce a concrete procedure to construct a new isotopy on the 2-torus generated by a time-dependent vector field, and the norms of the isotopy are controlled in terms of the norms of the initial diffeomorphism.

This problem of finding a canonically defined isotopy between diffeomorphisms with controlled norms, seemingly naive, is non-trivial even for a 2-torus. Given a sequence of diffeomorphisms of, say a 2-torus, they may be all conjugate to each other, or their conjugates converge (meaning they are the same or almost the same in some coordinate systems). This is one of the main motivations for the above mentioned problem. One method of detecting this is to find a conjugation-invariant norm which grows to infinity on the diffeomorphims in question. Another approach is finding a canonical (normal) form, which is what we are trying in our paper.

Given a horizontal geodesic on the standard torus $S^1 \times S^1$, the initial diffeomorphism can send it to a complicated embedded closed curve if the diffeomorphism has large norms. One of the most efficient ways to straighten this complicated curve is to apply the curve shortening flow along which every point moves with the velocity that is the geodesic curvature. This geometric flow behaves like a heat equation keeping all the evolving curves embedded, and singularity does not occur unless the flow shrinks to a point ([6][7][8]). The latter case does not happen, since our diffeomorphism is homotopic to the identity and hence the curve in question is not contractible. The flow at first straightens the curve rapidly, though possibly introducing points with very large but finite curvature, and slows down once the

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curvature becomes small. When the curvature is sufficiently small the curve becomes simply a graph, in which case straightening the curve is a simple task.

Consider the whole family of horizontal geodesics on the standard torus $S^1 \times S^1$. Then apply the curve shortening flow to their images* via the diffeomorphism F all at once up to an explicit time $T_0 = (||F||_{C^1} + 1)^2$. By a compactness argument (Proposition 2.2), the curvatures are uniformly bounded for all such flows up to time T_0. Denote the uniform curvature bound by K.

Main Theorem. *Suppose F is a $C^{m,\alpha}$ $(m \geqslant 3,\ \alpha \in (0,1))$ diffeomorphism of the 2-torus and it is homotopic to the identity. Then there is a $C^{[\frac{m-1}{2}],\alpha}$ isotopy (generated by a time-dependent vector field) between F and the identity, and the isotopy restricted to each fixed time is a $C^{m-2,\alpha}$ diffeomorphism. Furthermore the norms of the isotopy are explicitly controlled depending only on $||F||_{C^{m,\alpha}}, ||F^{-1}||_{C^1}, K, m, \alpha$, where K is the uniform curvature bound depending only on $||F||_{C^{3,\alpha}}, ||F^{-1}||_{C^1}$.*

Remark. *It turns out to be difficult to explicitly estimate the uniform curvature bound K only in terms of the initial diffeomorphism. Consider a closed curve which winds many times within a small region. In a short time part of the curve can shrink rapidly causing large curvature to appear. However once the curvature becomes too large, it cannot keep growing much more otherwise a singularity will form. The curve shortening flow is one of the fastest way to decrease large curvatures, but it appears to be too fast as far as we are concerned.*

We believe that the right characteristic which can be explicitly controlled along a curve shortening procedure is the largest radius $r(\gamma)$ such that one can touch a curve γ by a ball of radius r at every point and either side so that the interior of the ball does not intersect the curve, historically named the reach. We think we can prove the following fact: On the space of homotopically non-trivial curves γ with $r(\gamma) \geqslant r_0$ on a 2-torus, the length functional has no critical points other than closed geodesics. (In the plane, the critical points are circles of radius r_0.) This allows us to construct a curve shortening procedure with explicitly controlled curvature $k \leqslant 1/r(\gamma_0)$, where γ_0 is the inital curve. The problem is that the procedure we have in mind is only $C^{1,1}$.

The problem came from a discussion between D. Burago and L. Polterovich. We apply the curve shortening flow to all the embedded closed curves which are the images of the horizontal geodesics via the diffeomorphism F at once, and they become a family of horizontal geodesics as time goes to infinity by Grayson's Theorem (Theorem 2.1). We stop the curve shortening flow at some time after which all the curves become graphs. Then we can simply move them back to the original family of horizontal geodesics via the "height function", up to a reparametrization. The details can be found in Section 3.

The Main Theorem provides a control on the isotopy between two homotopically trivial diffeomorphisms on the standard 2-torus $S^1 \times S^1$. Without loss of generality,

*By image we mean the curve with parametrization coming from the composition of the diffeomorphism and the natural parametrization of the horizontal geodesic.

we assume that one of the diffeomorphisms is the identity. The theorem also applies to homotopically non-trivial orientation-preserving diffeomorphisms because of the classical fundamental theorem of Dehn that the group of isotopy classes of orientation-preserving diffeomorphisms of $S^1 \times S^1$ is isometric to $SL(2, \mathbb{Z})$ which is generated by two Dehn twists ([5]), in which case similar control can be obtained in the same way. We prove the Main Theorem for the standard 2-torus $S^1 \times S^1$. For a general 2-torus, there exists a diffeomorphism from it to the standard 2-torus. So all the distortions to the isotopy caused by the non-standard metric are controlled by the norms of that diffeomorphism. Our isotopy exists if the initial diffeomorphism has lower regularity and one can expect a Lipschitz estimate on the isotopy.

We do not know if similar control on the isotopy can be obtained for S^2. Motivated by works of A. Nabutovsky and S. Weinberger ([10][11][12]), we have serious doubts if it is possible for n-tori for $n \geqslant 5$. The main result of [10] implies that diffeomorphisms of S^n ($n \geqslant 5$) embedded in $\mathbb{R}^{n+1}$ cannot be extended to the ambient space with algorithmly controllable norms, and we think our work is closely related to the possible extendability for the base dimension.

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2. Preliminaries

Let's review a few things about the standard curve shortening flow on a smooth Riemannian surface. The curve shortening flow is a family of closed curves $\gamma(y,t)$: $S^1 \times [0, T_{max}) \to M$ evolving according to the equation

$$\partial_t \gamma(y,t) = k(y,t)\mathbf{N}(y,t) \tag{2.1}$$

with initial condition $\gamma(y,0) = \gamma_0(y)$, where $\mathbf{N}(y,t)$ is unit normal vector with respect to the time t curve $\gamma(\cdot,t)$ at the point $\gamma(y,t)$. The length of time t curve $\gamma(\cdot,t)$ (or γ_t) is denoted by $L(\gamma_t)$ and satisfies the formula

$$\frac{d}{dt} L(\gamma_t) = -\int_{\gamma_t} k^2(s,t)ds, \tag{2.2}$$

where s is the arc-length parameter. This shows the length of a family of evolving curves is monotone decreasing in time.
The finiteness of the maximum time of existence T_{max} completely determines the geometric behavior near the maximum time by the following Grayson's Theorem.

Theorem 2.1 ([8]). *If the initial closed curve is smooth and embedded, the curve shortening flow either converges to a point in finite time, or converges to a closed geodesic in C^∞ norm in infinite time and never develops singularities.*

From now on, consider the Riemannian surface to be the standard 2-torus $\mathbb{T}^2 = S^1 \times S^1$. The diffeomorphism F in question is of $C^{m,\alpha}$ ($m \geqslant 3$, $\alpha \in (0,1)$). We only consider the initial curve γ_0 being the image of a horizontal geodesic via the diffeomorphism F which is homotopic to the identity, and we know that this family of evolving curves γ_t does not shrink to a point and instead it must converge to a horizontal geodesic in infinite time by Theorem 2.1. Since our family of initial curves are uniformly bounded in certain norms by the norms of the diffeomorphism, we can ask for a uniform curvature bound under the evolution within any finite fixed time T.

Proposition 2.2. *Up to any finite fixed time T, the curvatures are uniformly bounded for all families of evolving curves with the initial curves being the images of horizontal geodesics via the $C^{m,\alpha}$ diffeomorphism F, and the bound depends only on $||F||_{C^{3,\alpha}}, ||F^{-1}||_{C^1}, T$. Denote the uniform curvature bound by K.*

Proof. For a fixed diffeomorphism F, the boundedness is due to a trivial compactness argument. Indeed, the space of the images of all horizontal geodesics via the fixed diffeomorphism is compact, and by the continuous dependence of the curve shortening flow on the initial condition, the curvatures are uniformly bounded up to time T.

However, proving the dependence of the bound requires a little stronger argument. We range the diffeomorphism over all $C^{m,\alpha}$ diffeomorphisms with fixed $C^{3,\alpha}$ norm and C^1 norm of the inverse, and consider all the curve shortening flows with initial curves being the image of horizontal geodesics via all diffeomorphisms in this class. We need to prove that the curvatures are uniformly bounded for all such flows up to any finite fixed time T.

We argue by contradiction. Suppose unbounded, so there exists a sequence of curve shortening flows whose curvatures can get arbitrarily large. Take the sequence of their initial curves and by compactness there exists a subsequence converging to a limit curve in C^3. Our choice of the class of diffeomorphisms along with a simple compactness reasoning are sufficient to guarantee that the limit curve is embedded, regular, and not homotopic to a point. So the limit curve under the evolution exists for all times and has bounded curvature by Theorem 2.1. By the continuous dependence of the curve shortening flow on the initial condition([4]), if the initial curve is close to the limit curve in C^3, their curvatures stay close to each other under the evolution in finite time. This is a contradiction to the assumption that the curvatures in the assumptive sequence of flows can get arbitrarily large. □

3. Regularity

In this section, we prove the main regularity result and consequently the main theorem. First, we need the following lemma which uniformly determines an explicit time after which the curves become graphs. We restrict our attention to the evolving curves γ_t with the initial curve γ_0 being the image of some horizontal geodesic via the diffeomorphism F.

Lemma 3.1. *Starting from $T_0 = (||F||_{C^1} + 1)^2$, all the curves under the evolution are graphs.*

Proof. Suppose T_0 is the earliest time when γ_t becomes a graph. So for any $t < T_0$, γ_t has a segment β_t which has integral of curvature at least $\pi/2$. By Cauchy-Schwartz inequality,

$$L(\gamma_t)\int_{\gamma_t} k^2 ds > (\int_{\beta_t} kds)^2 \geqslant \frac{\pi^2}{4}.$$

Since the length of γ_t is decreasing in time, $L(\gamma_t) \leqslant L(\gamma_0) \leqslant ||F||_{C^1} + 1$. By equation (2.2) for $t < T_0$,

$$\frac{d}{dt}L(t) = -\int_{\gamma_t} k^2 ds < -\frac{\pi^2}{4L(\gamma_t)} < -\frac{1}{||F||_{C^1}+1}.$$

This implies at time T_0, $L(\gamma_{T_0}) < L(\gamma_0) - \frac{1}{||F||_{C^1}+1}T_0$. Therefore

$$T_0 < L(\gamma_0)(||F||_{C^1} + 1) \leqslant (||F||_{C^1} + 1)^2.$$

Once a curve becomes a graph, its vertical translations do not intersect. Thus their images under the evolution do not intersect either due to the maximum principle. Hence the curves remain graphs forever. □

Denote by $\Phi(x,t) : \mathbb{T}^2 \times \mathbb{R}_+ \to \mathbb{T}^2$ the flow generated by the evolution, namely by the time-dependent vector field $k\mathbf{N}$. The flow is well defined due to the maximum principle. In the end we are going to stop the flow at the time T_0 which is determined in Lemma 3.1, while for the moment we state the following regularity result within any finite fixed time T.

Proposition 3.2. *For each $t \in [0,T]$, $\Phi(\cdot,t)$ is a $C^{m-2,\alpha}$ diffeomorphism, and $\Phi(x,t)$ is a $C^{[\frac{m-1}{2}],\alpha}$ map on $\mathbb{T}^2 \times [0,T]$. The norms are explicitly controlled depending only on $||F||_{C^{m,\alpha}}, ||F^{-1}||_{C^1}, K, T, m, \alpha$, where K is the uniform curvature bound described in Proposition* 2.2.

Proof. First for any fixed time $t \in [0,T]$, $\Phi(\cdot,t)$ is a homeomorphism. Indeed, the map $\Phi(\cdot,t)$ is injective by the maximum principle and continuous by the continuous dependence of the curve shortening flow on the initial condition. Due to the Invariance of Domain Theorem, the image of $\Phi(t)$ is open and $\Phi(t)$ is a homeomorphism from $\mathbb{T}^2$ to its image. The image is on the other hand compact, hence closed. Thus $\Phi(t)$ is surjective and is a homeomorphism of $\mathbb{T}^2$.

Now we discuss the differentiability of the flow. There are three directions to consider: two spatial directions and one time direction. Two of them, tangential direction of the curves and time direction, are smooth among themselves due to the analyticity of the solution of parabolic equation at positive times, and their norms are uniformly bounded since all derivatives of curvatures grow at most exponentially in finite time with the exponent depending on the uniform curvature bound K.

The only problem is the other spatial direction, namely across different families of flows. This is essentially the smooth dependence on the initial condition. It is known that the solution of the curve shortening flow exhibits C^r-smooth dependence on any parameter if the equation depends C^r smoothly on that parameter([1]). In our case, we can absorb the initial condition into a parameter, carry out a similar argument as the proof of the smooth dependence on parameters, and obtain an explicit estimate of the regularity.

Consider the equation of the curve shortening flow

$$\frac{\partial \gamma^\epsilon(y,t)}{\partial t} = k(\gamma^\epsilon)\mathbf{N}(\gamma^\epsilon), \tag{3.1}$$

with varying initial conditions $\gamma^\epsilon(0) = F(\tilde{\gamma}_0 + \epsilon \mathbf{h})$ for sufficiently small ϵ, where F is the initial $C^{m,\alpha}$ diffeomorphism, $\tilde{\gamma}_0$ is a fixed horizontal geodesic and $\mathbf{h}$ is the unit vertical vector. Here ϵ is the parameter representing the unclear spatial direction in question. We aim for the differentiabiliy with respect to the parameter ϵ.

First, we reduce the equation for curves (3.1) to equation for functions. For sufficiently small ϵ, all the initial curves $\gamma^\epsilon(0)$ lie in the normal coordinate system determined by equidistant curves around the fixed curve $\gamma^0(0) = F(\tilde{\gamma}_0)$. Hence the local solutions in sufficiently small time also lie in the same coordinate system. We define a C^m function u for a C^m curve γ within the coordinate system to be the distance function of the curve γ from the fixed curve $F(\tilde{\gamma}_0)$. In this way the local solution curves $\gamma^\epsilon(y,t)$ are represented by functions $u(y,t,\epsilon)$, so are their curvatures and unit normal vectors. Thus the equation for curves (3.1) is reduced to an equation for functions $u(y,t,\epsilon)$. By straightforward calculations mostly identical to the ones in [1], one can obtain the following evolution equation satisfied by $u = u(y,t,\epsilon)$ for sufficiently small ϵ, t:

$$u_t = G(y, u, u_y, u_{yy}), \tag{3.2}$$

where G is a nonlinear function of the four arguments, with the initial condition $u(y,0,\epsilon) = dist(\gamma^\epsilon(y,0), \gamma^0(y,0))$ within the said normal coordinate system. We denote u_y by p and u_{yy} by q. The function G satisfies the following properties:
(1) $\frac{\partial G}{\partial q}$ is positive, which implies the equation is parabolic;
(2) The function G is a $C^{m-2,\alpha}$ function with respect to all arguments;
(3) More precisely, the $C^{m-3,\alpha}$ norms of all the partial derivatives of G are uniformly bounded explicitly depending only on $||F||_{C^{m,\alpha}}, ||F^{-1}||_{C^1}, K, T, m$.

Now we absorb the varying initial conditions into the equation (3.2) by subtracting $u(y,0,\epsilon)$, and we obtain the equation of $\tilde{u}(y,t,\epsilon) = u(y,t,\epsilon) - u(y,0,\epsilon)$ with vanishing initial condition, represented by another nonlinear function $\tilde{G}$:

$$\tilde{u}_t = \tilde{G}(y, \epsilon, \tilde{u}, \tilde{p}, \tilde{q}),$$

where $\tilde{p}$, $\tilde{q}$ denote the first and second derivative of $\tilde{u}$ with respect to y respectively. The parameter ϵ appears in terms of the initial condition $u(y,0,\epsilon)$ and its derivatives with respect to y up to the second order, which implies $\tilde{G}$ is $C^{m-2,\alpha}$-smooth with respect to ϵ. It follows that all the properties of G hold for $\tilde{G}$.

Denote the Fréchet derivative of $\tilde{G}$ at $\tilde{u}$ of any v by $\big(d\tilde{G}(\tilde{u})\big)v = \tilde{G}_{\tilde{q}} v_{yy} + \tilde{G}_{\tilde{p}} v_y + \tilde{G}_{\tilde{u}} v$. Thanks to the properties of $\tilde{G}$, by a standard argument (e.g. [2]), the equation at $\tilde{u}$: $\big(\partial_t - d\tilde{G}(\tilde{u})\big)v = f$ has a solution $v \in C^{m-1,\alpha}$ with the initial condition $v(0) = 0$ for every $f \in C^{m-3,\alpha}$, which means $\partial_t - d\tilde{G}(\tilde{u})$ is invertible. The Implicit Function Theorem in Banach spaces implies that $\tilde{u}$ is C^{m-2}-smooth with respect to y and ϵ, so is u. Equipped with the differentiability of u, we can estimate its derivatives directly from the equation (3.2).

Differentiate the equation (3.2) with respect to ϵ, and we get the equation satisfied by u_ϵ:

$$u_{\epsilon t} = \frac{\partial G}{\partial q} u_{\epsilon yy} + \frac{\partial G}{\partial p} u_{\epsilon y} + \frac{\partial G}{\partial u} u_\epsilon,$$

with the initial condition to be $u_\epsilon(y, 0, \epsilon)$ which is $C^{m-1,\alpha}$-smooth with respect to y. By the regularity estimate of the linear parabolic equation (Theorem 5.1.9 in [9]), up to a sufficiently small time t_0,

$$\begin{aligned} ||\frac{\partial u}{\partial \epsilon}(y, t, \epsilon)||_{C^{m-1,\alpha}(S^1\times[0,t_0])} &\leqslant e^{C_1 t_0} ||\frac{\partial u}{\partial \epsilon}(y, 0, \epsilon)||_{C^{m-1,\alpha}(S^1\times[0,t_0])} \\ &\leqslant C(||F||_{C^{m,\alpha}}, ||F^{-1}||_{C^1}, K, T, m, \alpha) e^{C_1 t_0}, \end{aligned}$$

where $C_1 = C_1(||F||_{C^{m,\alpha}}, ||F^{-1}||_{C^1}, K, T, m, \alpha)$. Here we freeze the parameter ϵ and consider differentiation only with respect to y. Similar estimate also holds for $||\frac{\partial u}{\partial y}||_{C^{m-1,\alpha}}$. By a standard bootstrapping argument, we arrive at the following estimate for higher order derivatives with respect to ϵ up to the order $m-2$:

(3.3)

$$||\frac{\partial^{(i+j)} u}{\partial y^i \partial \epsilon^j}||_{C^{0,\alpha}} \leqslant C(||F||_{C^{m,\alpha}}, ||F^{-1}||_{C^1}, K, T, t_0, m, \alpha), \quad \text{for } i+j \leqslant m \text{ and } j \leqslant m-2.$$

Since we know in advance that our flow exists for all times, it is not difficult to check the same procedure can be continued from t_0, primarily because the regularity estimate (3.3) gives sufficient regularity at time t_0 to repeat the procedure. Thus with some care in tracing time-involved terms at each step, one can extend the regularity estimate to any finite fixed time T. The estimate on u implies the estimate on the original curve γ^ϵ:

(3.4)

$$||\frac{\partial^{(i+j)} \gamma^\epsilon}{\partial y^i \partial \epsilon^j}||_{C^{0,\alpha}} \leqslant C(||F||_{C^{m,\alpha}}, ||F^{-1}||_{C^1}, K, T, m, \alpha), \quad \text{for } i+j \leqslant m \text{ and } j \leqslant m-2.$$

As a consequence, we also get the $C^{m-2,\alpha}$ differentiability of the unit tangent vector and unit normal vector of the original curve γ^ϵ with respect to ϵ.

The flow is $C^{m-2,\alpha}$ in spatial directions with respect to the parameters y, ϵ due to (3.4), consequently with respect to the original spatial parameter $x \in \mathbb{T}^2$ at time 0 determined by the initial diffeomorphism F. The only issue is the continuity of time derivatives in the ϵ direction. Since we are dealing with a second order equation (3.2), the regularity in the time direction is given by the regularity of the second derivative with respect to the tangential direction. It follows that every time taking a time derivative decreases the regularity in the ϵ direction by 2 in view of the regularity estimate (3.4). Thus one can only take time derivatives at most $[\frac{m-1}{2}]$ times before losing the differentiability in the ϵ direction. Therefore the total regularity of the flow will be reduced to $[\frac{m-1}{2}]$.

Up to this point we have proved that for each fixed time $t \in [0, T]$, $\Phi(\cdot, t)$ is a $C^{m-2,\alpha}$ homeomorphism. To argue it is a diffeomorphism, by the Inverse Function Theorem it suffices to prove that the differential $d\Phi(\cdot, t)$ is nonsingular everywhere. Again we look at the two spatial directions. Along the tangential direction of the curve, the length of the velocity is uniformly bounded away from 0 in finite time, due to the evolution equation satisfied by the velocity $\partial_t|\gamma_y| = -k^2|\gamma_y|$ along with the uniform curvature bound K. The non-singularity of the differential $d\Phi(\cdot,t)$ across different families of flows follows from the fact that the distance between two curves is nondecreasing in time via the curve shortening flow ([6]). Furthermore one can show that the length of the differential $d\Phi(\cdot,t)$ acting on any unit vector is bounded away from 0 depending only on $||F||_{C^1}, ||F^{-1}||_{C^1}, K, T$, which implies the differential is nonsingular. By the Inverse Function Theorem, the inverse $\Phi(\cdot,t)^{-1}$ is a C^{m-2} homeomorphism and hence $\Phi(\cdot,t)$ is a diffeomorphism. □

Now we prove the main theorem. First let us closely examine what happens to one single curve. Take a horizontal geodesic, and after applying the initial diffeomorphism F and the curve shortening flow up to time $T_0 = (||F||_{C^1}+1)^2$, we get a closed curve which is a graph with respect to the original horizontal geodesic by Lemma 3.1. However the parametrization of the curve is distorted, and we apply a second procedure of reparametrization to make it align with the natural parametrization of the horizontal geodesic. Then we apply a third procedure to move the curve back to the horizontal geodesic via the "height function". The details are discussed below. Although it is straightforward to handle one single curve, in our case we need to apply these procedures to all the curves at once, and to recover the regularity we need some consistencies when applying these procedures across different curves, which is provided by Proposition 3.2 specifically the regularity estimate (3.4).

We discuss the second procedure of reparametrizations here. The composition $\Phi_1 = \Phi(\cdot,T_0)\circ F$ is a $C^{m-2,\alpha}$ diffeomorphism of $\mathbb{T}^2$ by Proposition 3.2 and it is homotopic to the identity. This diffeomorphism Φ_1 maps the family of horizontal geodesics to a family of graphs. For a fixed horizontal geodesic, we consider a map f which sends the fixed horizontal geodesic to the vertical projection of its image (graph) via the diffemorphism Φ_1 onto itself. This map f is a diffeomorphism of S^1 and it is homotopically trivial. Hence the lift $\tilde{f}$ of the diffeomorphism f of S^1 on the universal cover $\mathbb{R}/\mathbb{Z}$ of S^1 has degree 1, i.e. $\tilde{f}(z+1) = \tilde{f}(z)+1$ for $z \in \mathbb{R}$. Thus the isotopy on the universal cover $\tilde{\phi}(z,t) = (1-t)\tilde{f}(z) + tz$ on $\mathbb{R}\times[0,1]$ between $\tilde{f}$ and the identity (of $\mathbb{R}$) descends to an isotopy on S^1 between f and the identity (of S^1). This isotopy on S^1 gives a reparametrization for a graph. Apply this construction to all the horizontal geodesics, and we obtain a reparametrization for the whole family of graphs, denoted by $\Phi_2(x,t)$ on $\mathbb{T}^2\times[0,1]$. Due to Proposition 3.2, this flow of reparametrization Φ_2 is $C^{m-2,\alpha}$. To glue the flow Φ_2 and the curve shortening flow together, we need modifications to both flows in order to preserve the regularity. Take a suitable smooth function $f_1 : [T_0, 4T_0] \to [T_0, 4T_0]$ satisfying that $f_1(t) = t$ on $[T_0, 2T_0]$, $f_1 = 4T_0$ on $[3T_0, 4T_0]$ and f_1 is strictly increasing on $[2T_0, 3T_0]$. Extend the curve shortening flow up to time $4T_0$, and within $[T_0, 4T_0]$ modify the flow as $\tilde{\Phi}(t) = \Phi(f_1(t))$. The norm of such modified flow is enlarged by a scale factor of a constant depending only on m. One can glue this modified curve shortening flow with a similarly modified flow of reparametrization without affecting the regularity. On the level of vector fields, this is simply the standard procedure to glue two time-dependent vector fields together along time by multiplying smooth functions of time vanishing in the gluing region.

Then we apply a third procedure to move the curves (graphs) back to the horizontal geodesics. The composition of the first two procedures $\Phi_2(\cdot, 1) \circ \Phi_1$ (properly glued) is a homotopically trivial $C^{m-2,\alpha}$ diffeomorphism of $\mathbb{T}^2$, and its restriction onto a fixed vertical geodesic defines a homotopically trivial diffeomorphism of S^1. We can simply repeat the construction of the second procedure to find an isotopy between this diffeomorphism of S^1 and the identity. Apply the construction to the whole family of vertical geodesics, and we obtain an isotopy on $\mathbb{T}^2$ between $\Phi_2(1) \circ \Phi_1$ and the identity. Again this isotopy is $C^{m-2,\alpha}$. One can similarly glue this isotopy with the first two procedures while preserving the regularity. The main theorem is proved.

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